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Solutions structures for some systems of fractional difference equations

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Abstract: It is a well-known fact that the majority of rational difference equations cannot be solved theoretically. As a result, some scientific experts use manual iterations to obtain the exact solutions of some of these equations. In this paper, we obtain the fractional solutions of the following systems of difference equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1-x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\pm 1\pm y_{n-1}x_{n-3}\right)},\quad n=0,1,...,$$

where the initial data $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are arbitrary non-zero real numbers. All solutions will be depicted under specific initial conditions.

Keywords: System of recursive equations, difference equations, equilibrium point, local stability, qualitative behaviour of solutions, periodicity.

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1. Introduction

The theory of discrete dynamical systems of difference equations has been utilized to study natural phenomena which change over discrete time. A massive number of researchers investigated various real life problems that occur in population dynamics, genetics in biology, engineering, queuing problems, electrical networks, physics, economics, etc [1]. The long-term behaviours of such problems have been recently discussed by some scholars. For example, in [2] Asiri *et al.* explained the periodic solutions of the following system of difference equations:

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm 1\pm x_{n-2}y_{n-1}x_n}.$$

Cinar [3] explored the periodicity of non-negative solutions of a system of fractional difference equations given by the form:

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

Din [4] analyzed and obtained the equilibrium points, local asymptotic stability and global behaviour of the fixed points of Lotka-Volterra model which is illustrated by the system:

$$x_{n+1}=\frac{\alpha x_n-\beta x_n y_n}{1+\gamma x_n},\quad y_{n+1}=\frac{\delta y_n+\epsilon x_n y_n}{1+\eta y_n}.$$

El-Metwally *et al.* [5] found the solutions and the periodicity of the following third order system of rational recursive equations:

$$x_{n+1}=\frac{y_{n-2}}{-1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm 1\pm x_{n-2}y_{n-1}x_n}.$$

In [6], Elsayed presented the solutions of the following system of second order difference equations:

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 \pm x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1}x_n}.$$

Elsayed and Alzubaidi [7] found the solutions of following systems of rational difference equations:

$$x_{n+1} = \frac{y_{n-8}}{1 + y_{n-2}x_{n-5}y_{n-8}}, \quad y_{n+1} = \frac{x_{n-8}}{\pm 1 \pm x_{n-2}y_{n-5}x_{n-8}}.$$

Gű mű ş and Őcalan [8] explored the positive solutions of the systems:

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}.$$

Kurbanli *et al.* [9] obtained the solutions of the following system of difference equations:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

In [10], Touafek *et al.* discovered the periodicity and solution of the system:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}.$$

The author in [11] examined the dynamics of the following system of recursive equations:

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}.$$

For more information about basic theory and qualitative behaviour of dynamical systems of difference equations, one can see references [12–27].
This work aims to present the solutions of discrete dynamical systems of difference equations which are given by

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1 - x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\pm 1 \pm y_{n-1}x_{n-3}\right)}, \quad n = 0, 1, \ldots,$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are required to be real numbers.

2. Main Results

2.1. First system $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(-1-x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(1+y_{n-1}x_{n-3})}$

This subsections is devoted for the solutions of the following system of recursive equations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1 - x_{n-1}y_{n-3}\right)}, \; y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(1 + y_{n-1}x_{n-3}\right)}, \; n = 0,\ 1, \ldots. \tag{1}$$

The initial values of this system are required to be arbitrary real numbers.

Theorem 1. *Let $\{x_n, y_n\}$ be a solution to system (1) and let $x_{-3} = a,\ x_{-2} = b,\ x_{-1} = c,\ x_0 = d,\ y_{-3} = \alpha,\ y_{-2} = \beta,\ y_{-1} = \gamma$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ \ldots,$ we have*

$$x_{4n-3} = \frac{(-1)^n c^n \alpha^n \prod_{i=0}^{n-1} \left[(2i)\, a\gamma + 1\right]}{a^{n-1}\gamma^n \left(c\alpha + 1\right)^n}, \quad x_{4n-2} = \frac{(-1)^n d^n \beta^n \prod_{i=0}^{n-1} \left[(2i)\, b\omega + 1\right]}{b^{n-1}\omega^n \left(d\beta + 1\right)^n},$$

$$x_{4n-1} = \frac{c^{n+1}\alpha^n \prod_{i=0}^{n-1} \left[(2i+1)\, a\gamma + 1\right]}{a^n\gamma^n}, \quad x_{4n} = \frac{d^{n+1}\beta^n \prod_{i=0}^{n-1} \left[(2i+1)\, b\omega + 1\right]}{b^n\omega^n},$$

and

$$y_{4n-3} = \frac{a^n\gamma^n}{c^n\alpha^{n-1}\prod_{i=0}^{n-1}[(2i+1)a\gamma+1]}, \quad y_{4n-2} = \frac{b^n\omega^n}{d^n\beta^{n-1}\prod_{i=0}^{n-1}[(2i+1)b\omega+1]},$$
$$y_{4n-1} = \frac{(-1)^n a^n\gamma^{n+1}(c\alpha+1)^n}{c^n\alpha^n\prod_{i=0}^{n-1}[(2i+2)a\gamma+1]}, \quad y_{4n} = \frac{(-1)^n b^n\omega^{n+1}(d\beta+1)^n}{d^n\beta^n\prod_{i=0}^{n-1}[(2i+2)b\omega+1]}.$$

Proof. It is easy to see that the results hold for $n = 0$. Next, we assume that $n > 1$ and suppose that the solutions hold for $n-1$. That is

$$x_{4n-7} = \frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}, \quad x_{4n-6} = \frac{(-1)^{n-1}d^{n-1}\beta^{n-1}\prod_{i=0}^{n-2}[(2i)b\omega+1]}{b^{n-2}\omega^{n-1}(d\beta+1)^{n-1}},$$
$$x_{4n-5} = \frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}}, \quad x_{4n-4} = \frac{d^n\beta^{n-1}\prod_{i=0}^{n-2}[(2i+1)b\omega+1]}{b^{n-1}\omega^{n-1}}$$

and

$$y_{4n-7} = \frac{a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}, \quad y_{4n-6} = \frac{b^{n-1}\omega^{n-1}}{d^{n-1}\beta^{n-2}\prod_{i=0}^{n-2}[(2i+1)b\omega+1]},$$
$$y_{4n-5} = \frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}, \quad y_{4n-4} = \frac{(-1)^{n-1}b^{n-1}\omega^n(d\beta+1)^{n-1}}{d^{n-1}\beta^{n-1}\prod_{i=0}^{n-2}[(2i+2)b\omega+1]}.$$

Following this, system (1) gives us

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}(-1-x_{4n-5}y_{4n-7})} \\ &= \frac{\frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}}\frac{a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}}{\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\left[-1-\frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}}\frac{a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}\right]} \\ &= \frac{-(-1)^{-n+1}c^n\alpha^n\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}{a^{n-1}\gamma^n(c\alpha+1)^{n-1}[1+c\alpha]} = \frac{(-1)^n c^n\alpha^n\prod_{i=0}^{n-1}[(2i)a\gamma+1]}{a^{n-1}\gamma^n(c\alpha+1)^n}. \end{aligned}$$

Moreover, one can observe from system (1) that

$$\begin{aligned} y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}[1+y_{4n-5}x_{4n-7}]} \\ &= \frac{\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}}{\frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}}\left[1+\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}\right]} \end{aligned}$$

$$= \frac{\frac{a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}}{\frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}}\left[1+\frac{a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\right]}$$

$$= \frac{a\gamma\prod_{i=0}^{n-2}\left[(2i)\,a\gamma+1\right]a^{n-1}\gamma^{n-1}}{c^n\alpha^{n-1}\prod_{i=0}^{n-2}\left[(2i+1)\,a\gamma+1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\,a\gamma+1\right]+a\gamma\prod_{i=0}^{n-2}\left[(2i)\,a\gamma+1\right]\right]}$$

$$= \frac{a^n\gamma^n}{c^n\alpha^{n-1}\prod_{i=0}^{n-1}\left[(2i+1)\,a\gamma+1\right]}.$$

Similarly, other results can be proved. The proof has been completed. □

2.2. Second System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1-x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(-1-y_{n-1}x_{n-3})}$

In this section, we obtained the solutions of the following system of differential equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1-x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1-y_{n-1}x_{n-3}\right)}. \tag{2}$$

The initial values of system (2) are required to be arbitrary real numbers.

Theorem 2. *Let $\{x_n, y_n\}$ be a solution to system (2) and assume that $x_{-3}=a,\ x_{-2}=b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=\alpha,\ y_{-2}=\beta,\ y_{-1}=\gamma$ and $y_0=\omega$. Then, for $n=0,\ 1,\ \ldots$ we have*

$$x_{4n-3} = \frac{(-1)^n c^n\alpha^n}{a^{n-1}\gamma^n\left(c\alpha+1\right)^n},\quad x_{4n-2}=\frac{(-1)^n d^n\beta^n}{b^{n-1}\omega^n\left(d\beta+1\right)^n},$$
$$x_{4n-1} = \frac{(-1)^n c^{n+1}\alpha^n\left(a\gamma+1\right)^n}{a^n\gamma^n},\quad x_{4n}=\frac{(-1)^n d^{n+1}\beta^n\left(b\omega+1\right)^n}{b^n\omega^n}$$

and

$$y_{4n-3} = \frac{(-1)^n a^n\gamma^n}{c^n\alpha^{n-1}\left(a\gamma+1\right)^n},\quad y_{4n-2}=\frac{(-1)^n b^n\omega^n}{d^n\beta^{n-1}\left(b\omega+1\right)^n},$$
$$y_{4n-1} = \frac{(-1)^n a^n\gamma^{n+1}\left(c\alpha+1\right)^n}{c^n\alpha^n},\quad y_{4n}=\frac{(-1)^n b^n\omega^{n+1}\left(d\beta+1\right)^n}{d^n\beta^n}.$$

Proof. The solutions are verified for $n=0$. Next, we let $n>1$ and assume that the results hold for $n-1$. That is

$$x_{4n-7} = \frac{(-1)^{n-1} c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}\left(c\alpha+1\right)^{n-1}},\quad x_{4n-6}=\frac{(-1)^{n-1} d^{n-1}\beta^{n-1}}{b^{n-2}\omega^{n-1}\left(d\beta+1\right)^{n-1}},$$
$$x_{4n-5} = \frac{(-1)^{n-1} c^n\alpha^{n-1}\left(a\gamma+1\right)^{n-1}}{a^{n-1}\gamma^{n-1}},\quad x_{4n-4}=\frac{(-1)^{n-1} d^n\beta^{n-1}\left(b\omega+1\right)^{n-1}}{b^{n-1}\omega^{n-1}}$$

and

$$y_{4n-7} = \frac{(-1)^{n-1} a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\left(a\gamma+1\right)^{n-1}},\quad y_{4n-6}=\frac{(-1)^{n-1} b^{n-1}\omega^{n-1}}{d^{n-1}\beta^{n-2}\left(b\omega+1\right)^{n-1}},$$
$$y_{4n-5} = \frac{(-1)^{n-1} a^{n-1}\gamma^n\left(c\alpha+1\right)^{n-1}}{c^{n-1}\alpha^{n-1}},\quad y_{4n-4}=\frac{(-1)^{n-1} b^{n-1}\omega^n\left(d\beta+1\right)^{n-1}}{d^{n-1}\beta^{n-1}}.$$

Now, the first relation is given by

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(-1-x_{4n-5}y_{4n-7}\right)} \\ &= \frac{\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}}\frac{(-1)^{n-1}a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}(a\gamma+1)^{n-1}}}{\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}}\left[-1-\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}}\frac{(-1)^{n-1}a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}(a\gamma+1)^{n-1}}\right]} \\ &= \frac{-\left(-1\right)^{-n+1}c\alpha\; c^{n-1}\alpha^{n-1}}{a^{n-1}\gamma^n\left(c\alpha+1\right)^{n-1}\left[1+c\alpha\right]} = \frac{\left(-1\right)^n c^n\alpha^n}{a^{n-1}\gamma^n\left(c\alpha+1\right)^n}. \end{aligned}$$

Similarly, system (2) leads

$$\begin{aligned} y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(-1-y_{4n-5}x_{4n-7}\right)} \\ &= \frac{\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}}\frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}}{\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}}\left[-1-\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}}\frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}\right]} \\ &= \frac{-\left(-1\right)^{-n+1}a\gamma\; a^{n-1}\gamma^{n-1}}{c^n\alpha^{n-1}\left(a\gamma+1\right)^{n-1}\left[1+a\gamma\right]} = \frac{\left(-1\right)^n a^n\gamma^n}{c^n\alpha^{n-1}\left(a\gamma+1\right)^n}. \end{aligned}$$

Accordingly, the remaining relations of system (2) can be verified. Hence, this achieves the proof. □

2.3. Third System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1-x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})}$

In this subsection, the solutions of the following dynamic discrete system will be formulated:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1-x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(1-y_{n-1}x_{n-3}\right)},\tag{3}$$

where the initial values are required to be arbitrary real numbers.

Theorem 3. *Assume that* $\{x_n, y_n\}$ *is a solution to system (3) and suppose that* $x_{-3}=a,\ x_{-2}=b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=\alpha,\ y_{-2}=\beta,\ y_{-1}=\gamma$ *and* $y_0=\omega$. *Then, for* $n=0,\ 1,\ \ldots$ *we have*

$$\begin{aligned} x_{4n-3} &= \frac{c^n\alpha^n\prod_{i=0}^{n-1}\left[(2i)\,a\gamma-1\right]}{a^{n-1}\gamma^n\left(c\alpha+1\right)^n},\quad x_{4n-2}=\frac{d^n\beta^n\prod_{i=0}^{n-1}\left[(2i)\,b\omega-1\right]}{b^{n-1}\omega^n\left(d\beta+1\right)^n}, \\ x_{4n-1} &= \frac{\left(-1\right)^n c^{n+1}\alpha^n\prod_{i=0}^{n-1}\left[(2i+1)\,a\gamma-1\right]}{a^n\gamma^n},\quad x_{4n}=\frac{\left(-1\right)^n d^{n+1}\beta^n\prod_{i=0}^{n-1}\left[(2i+1)\,b\omega-1\right]}{b^n\omega^n} \end{aligned}$$

and

$$\begin{aligned} y_{4n-3} &= \frac{\left(-1\right)^n a^n\gamma^n}{c^n\alpha^{n-1}\prod_{i=0}^{n-1}\left[(2i+1)\,a\gamma-1\right]},\quad y_{4n-2}=\frac{\left(-1\right)^n b^n\omega^n}{d^n\beta^{n-1}\prod_{i=0}^{n-1}\left[(2i+1)\,b\omega-1\right]}, \\ y_{4n-1} &= \frac{a^n\gamma^{n+1}\left(c\alpha+1\right)^n}{c^n\alpha^n\prod_{i=0}^{n-1}\left[(2i+2)\,a\gamma-1\right]},\quad y_{4n}=\frac{b^n\omega^{n+1}\left(d\beta+1\right)^n}{d^n\beta^n\prod_{i=0}^{n-1}\left[(2i+2)\,b\omega-1\right]}. \end{aligned}$$

Proof. The solutions hold for $n=0$. Now, we suppose that $n>1$ and assume that the solutions hold for $n-1$. That is

$$x_{4n-7}=\frac{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}\left[(2i)\,a\gamma-1\right]}{a^{n-2}\gamma^{n-1}\left(c\alpha+1\right)^{n-1}},\quad x_{4n-6}=\frac{d^{n-1}\beta^{n-1}\prod_{i=0}^{n-2}\left[(2i)\,b\omega-1\right]}{b^{n-2}\omega^{n-1}\left(d\beta+1\right)^{n-1}},$$

$$x_{4n-5} = \frac{(-1)^{n-1} c^n \alpha^{n-1} \prod_{i=0}^{n-2} [(2i+1) a\gamma - 1]}{a^{n-1}\gamma^{n-1}}, \quad x_{4n-4} = \frac{(-1)^{n-1} d^n \beta^{n-1} \prod_{i=0}^{n-2} [(2i+1) b\omega - 1]}{b^{n-1}\omega^{n-1}}$$

and

$$y_{4n-7} = \frac{(-1)^{n-1} a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2} \prod_{i=0}^{n-2} [(2i+1) a\gamma - 1]}, \quad y_{4n-6} = \frac{(-1)^{n-1} b^{n-1}\omega^{n-1}}{d^{n-1}\beta^{n-2} \prod_{i=0}^{n-2} [(2i+1) b\omega - 1]},$$

$$y_{4n-5} = \frac{a^{n-1}\gamma^{n} (c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1} \prod_{i=0}^{n-2} [(2i+2) a\gamma - 1]}, \quad y_{4n-4} = \frac{b^{n-1}\omega^{n} (d\beta+1)^{n-1}}{d^{n-1}\beta^{n-1} \prod_{i=0}^{n-2} [(2i+2) b\omega - 1]}.$$

Next, it can be simply seen from system (3) that

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(-1 - x_{4n-5}y_{4n-7}\right)} \\ &= \frac{\frac{(-1)^{n-1} c^n \alpha^{n-1} \prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}} \frac{(-1)^{n-1} a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}}{\frac{a^{n-1}\gamma^n (c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\left[-1 - \frac{(-1)^{n-1} c^n \alpha^{n-1} \prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}} \frac{(-1)^{n-1} a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}\right]} \\ &= -\frac{c\alpha\, c^{n-1}\alpha^{n-1} \prod_{i=0}^{n-2} [(2i+2) a\gamma - 1]}{a^{n-1}\gamma^n (c\alpha+1)^{n-1} [1 + c\alpha]} = \frac{c^n \alpha^n \prod_{i=0}^{n-1} [(2i) a\gamma - 1]}{a^{n-1}\gamma^n (c\alpha+1)^n}. \end{aligned}$$

Furthermore, system (3) gives that

$$\begin{aligned} y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(1 - y_{4n-5}x_{4n-7}\right)} \\ &= \frac{\frac{a^{n-1}\gamma^n (c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]} \frac{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}}{\frac{(-1)^{n-1} c^n \alpha^{n-1} \prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}}\left[1 - \frac{a^{n-1}\gamma^n (c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]} \frac{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}\right]} \\ &= \frac{\frac{a\gamma \prod_{i=0}^{n-2}[(2i)a\gamma-1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}}{\frac{(-1)^{n-1} c^n \alpha^{n-1} \prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}}\left[1 - \frac{a\gamma \prod_{i=0}^{n-2}[(2i)a\gamma-1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\right]} \\ &= \frac{(-1)^{-n+1} a^n \gamma^n \prod_{i=0}^{n-2} [(2i) a\gamma - 1]}{c^n \alpha^{n-1} \prod_{i=0}^{n-2} [(2i+1) a\gamma - 1] \left[\prod_{i=0}^{n-2} [(2i+2) a\gamma - 1] - a\gamma \prod_{i=0}^{n-2} [(2i) a\gamma - 1]\right]} \\ &= \frac{(-1)^n a^n \gamma^n}{c^n \alpha^{n-1} \prod_{i=0}^{n-1} [(2i+1) a\gamma - 1]}. \end{aligned}$$

The remaining formulas can be shown in a similar way. □

2.4. Fourth System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1-x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

In the next theorem, we will discover the solutions of the following nonlinear system of difference equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1-x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1+y_{n-1}x_{n-3}\right)},\tag{4}$$

where the initial conditions are required to be arbitrary real numbers.

Theorem 4. *Assume that $\{x_n,y_n\}$ is a solution to system (4) and let $x_{-3}=a,\ x_{-2}=b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=\alpha,\ y_{-2}=\beta,\ y_{-1}=\gamma$ and $y_0=\omega$. Then, for $n=0,\ 1,\ ...$ we have*

$$\begin{aligned}x_{4n-3}&=\frac{(-1)^n c^n\alpha^n}{a^{n-1}\gamma^n\left(c\alpha+1\right)^n},\quad x_{4n-2}=\frac{(-1)^n d^n\beta^n}{b^{n-1}\omega^n\left(d\beta+1\right)^n},\\ x_{4n-1}&=\frac{c^{n+1}\alpha^n\left(a\gamma-1\right)^n}{a^n\gamma^n},\quad x_{4n}=\frac{d^{n+1}\beta^n\left(b\omega-1\right)^n}{b^n\omega^n}\end{aligned}$$

and

$$\begin{aligned}y_{4n-3}&=\frac{a^n\gamma^n}{c^n\alpha^{n-1}\left(a\gamma-1\right)^n},\quad y_{4n-2}=\frac{b^n\omega^n}{d^n\beta^{n-1}\left(b\omega-1\right)^n},\\ y_{4n-1}&=\frac{(-1)^n a^n\gamma^{n+1}\left(c\alpha+1\right)^n}{c^n\alpha^n},\quad y_{4n}=\frac{(-1)^n b^n\omega^{n+1}\left(d\beta+1\right)^n}{d^n\beta^n}.\end{aligned}$$

Proof. It is clear that the relations hold for $n=0$. Now, we suppose that $n>1$ and assume that the solutions hold for $n-1$. That is

$$\begin{aligned}x_{4n-7}&=\frac{(-1)^{n-1} c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}\left(c\alpha+1\right)^{n-1}},\quad x_{4n-6}=\frac{(-1)^{n-1} d^{n-1}\beta^{n-1}}{b^{n-2}\omega^{n-1}\left(d\beta+1\right)^{n-1}},\\ x_{4n-5}&=\frac{c^n\alpha^{n-1}\left(a\gamma-1\right)^{n-1}}{a^{n-1}\gamma^{n-1}},\quad x_{4n-4}=\frac{d^n\beta^{n-1}\left(b\omega-1\right)^{n-1}}{b^{n-1}\omega^{n-1}}\end{aligned}$$

and

$$\begin{aligned}y_{4n-7}&=\frac{a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}\left(a\gamma-1\right)^{n-1}},\quad y_{4n-6}=\frac{b^{n-1}\omega^{n-1}}{d^{n-1}\beta^{n-2}\left(b\omega-1\right)^{n-1}},\\ y_{4n-5}&=\frac{(-1)^{n-1} a^{n-1}\gamma^n\left(c\alpha+1\right)^{n-1}}{c^{n-1}\alpha^{n-1}},\quad y_{4n-4}=\frac{(-1)^{n-1} b^{n-1}\omega^n\left(d\beta+1\right)^{n-1}}{d^{n-1}\beta^{n-1}}.\end{aligned}$$

Now, it can be obviously observed from system (4) that

$$\begin{aligned}x_{4n-3}&=\frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(-1-x_{4n-5}y_{4n-7}\right)}\\ &=\frac{\frac{c^n\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}}\frac{a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}(a\gamma-1)^{n-1}}}{\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}}\left[-1-\frac{c^n\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}}\frac{a^{n-1}\gamma^{n-1}}{c^{n-1}\alpha^{n-2}(a\gamma-1)^{n-1}}\right]}\\ &=\frac{(-1)^{-n} c^n\alpha^n}{a^{n-1}\gamma^n\left(c\alpha+1\right)^{n-1}\left[1+c\alpha\right]}=\frac{(-1)^n c^n\alpha^n}{a^{n-1}\gamma^n\left(c\alpha+1\right)^n}.\end{aligned}$$

Similarly, one can obtain from system (4) that

$$\begin{aligned}y_{4n-3}&=\frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(-1+y_{4n-5}x_{4n-7}\right)}\\ &=\frac{\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}}\frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}}{\frac{c^n\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}}\left[-1+\frac{(-1)^{n-1}a^{n-1}\gamma^n(c\alpha+1)^{n-1}}{c^{n-1}\alpha^{n-1}}\frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}(c\alpha+1)^{n-1}}\right]}\end{aligned}$$

$$= \frac{a^n \gamma^n}{c^n \alpha^{n-1} (a\gamma - 1)^{n-1} [-1 + a\gamma]} = \frac{a^n \gamma^n}{c^n \alpha^{n-1} (a\gamma - 1)^n}.$$

Other relations can be likewise proved. Therefore, this completes our proof. □

2.5. Numerical Examples

This subsection is allocated to confirm our theoretical discussion by illustrating some numerical examples. These examples show the behaviour of the solutions of each system.

Example 1. In this example, we describe the behaviour of the solution of system (1). Our initial data has been taken as follows: $x_{-3} = 0.21,\ x_{-2} = -0.25,\ x_{-1} = -0.032,\ x_0 = 2,\ y_{-3} = 1.06,\ y_{-2} = -0.4,\ y_{-1} = -1.55$ and $y_0 = 0.082$. See Figure 1.

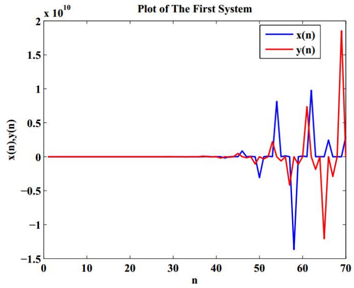

Figure 1. Model of the system.

Example 2. This example demonstrates the plot of system (2). The initial values are considered as follows: $x_{-3} = 0.22,\ x_{-2} = -0.25,\ x_{-1} = 0.2,\ x_0 = 0.6,\ y_{-3} = 1.03,\ y_{-2} = -0.43,\ y_{-1} = 1.5$ and $y_0 = 0.8$, as depicted in Figure 2.

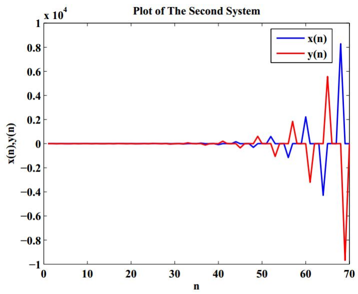

Figure 2. Model of the system.

Example 3. Here, we plot the curves of solutions of system (3). Figure 3 presents this behaviour under the following initial values: $x_{-3} = 0.03,\ x_{-2} = -0.2,\ x_{-1} = 10,\ x_0 = -0.05,\ y_{-3} = 2,\ y_{-2} = -4.1,\ y_{-1} = 1.4$ and $y_0 = 1.03$.

Example 4. Figure 4 illustrates the behaviour of the solution of system (4). The considered initial conditions in this example are given as follows: $x_{-3} = 0.02,\ x_{-2} = -0.2,\ x_{-1} = 0.21,\ x_0 = -0.03,\ y_{-3} = 0.1,\ y_{-2} = -4.9,\ y_{-1} = 1,\ y_0 = -1$.

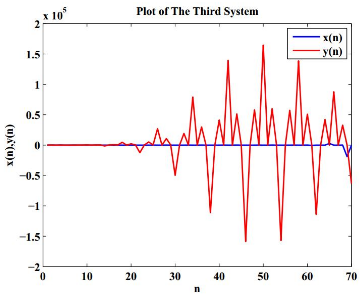

Figure 3. Model of the system.

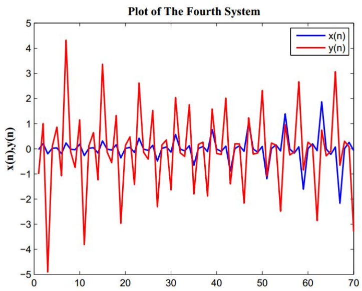

Figure 4. Model of the system.

3. Conclusion

This paper has been written to highlight the analytical and numerical solutions of four different systems of difference equations. In subsection 2.1, we have provided the solution of System 1 and illustrated its behaviour under specific conditions in Figure 1. Theorem 2 and Theorem 3 presented the exact solutions of System 2 and System 3, respectively. Finally, Figure 4 demonstrated the curve of the solution of System 4 which is given in Subsection 2.4.

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