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NECESSARY AND SUFFICIENT CONDITION FOR A SURFACE TO BE A SPHERE

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Abstract. Let S be a C^1-smooth closed connected surface in $\mathbb{R}^3$, the boundary of the domain D, $N = N_s$ be the unit outer normal to S at the point s, P be the normal section of D. A normal section is the intersection of D and the plane containing N. It is proved that if all the normal sections for a fixed N are discs, then S is a sphere. The converse statement is trivial.

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1. Introduction

Let S be a C^1-smooth closed connected surface in $\mathbb{R}^3$, the boundary of the domain D, $N = N_s$ be the unit outer normal to S at the point s. Throughout we assume that S satisfies these assumptions. Let P be the normal section of D. A normal section is the intersection of D and the plane containing N. Our result is the following:

Theorem 1.1. *If all the normal sections for a fixed N are discs, then S is a sphere. Conversely, if S is a sphere then all its normal sections are discs.*

There are several "characterizations" of the sphere in the literature. We will use the following.

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Lemma 1.2. *Let $r = r(p, q)$ be a parametric representation of S. If $[r(p, q), N_s] = 0$ for all $s = s(p, q)$ on S, then S is a sphere. Here $[r, N]$ is the vector product of two vectors.*

A proof of this result can be found in [1, 2]. For convenience of the reader a short proof of Lemma 1.2 is given in Section 2.

2. Proof

Theorem 1.1. Let $s \in S$ be a fixed point and P be one of the normal sections of D corresponding to N_s. By assumption, this section is a disc. Let O be its center and R be its radius. Rotate P about N_s. Each of the resulting normal sections is a disc of radius R centered at O. If $r = r(p, q)$ is a parametric representation of S then $[r, N] = 0$ for every point of S because each such point belongs to a boundary of a disc centered at O with radius R. From Lemma 1.2 it follows that S is a sphere. □

Lemma 1.2. One has $N = [r_p(p, q), r_q(p, q)]/|[r_p(p, q), r_q(p, q)]|$, where $[a, b]$ is the vector product of a and b, and $|a|$ is the length of the vector. Therefore $[r, N] = 0$ implies $[r, [r_p(p, q), r_q(p, q)]] = 0$ or $r_p(r, r_q) - r_q(r, r_p) = 0$, where (a, b) is the scalar product of two vectors. The vectors r_p and r_q are linearly independent since the surface S is smooth. Thus, $(r, r_q) = 0$ and $(r, r_p) = 0$. Consequently $(r, r) = const$, that is, S is a sphere. □

Competing Interests

The author declares that he has no competing interests.

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