Odd Lindley-Rayleigh Distribution: Its Properties and Applications to Simulated and Real Life Datasets

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Authors' contributions

This work was carried out in collaboration among all authors. Author TGI designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author SSA managed the analyses of the study. Author AAI managed the literature searches. All authors read and approved the final manuscript.

Article Information

Original Research Article

Abstract

This article develops an extension of the Rayleigh distribution with two parameters and greater flexibility which is an improvement over Lindley distribution, Rayleigh distribution and other generalizations of the Rayleigh distribution. The new model is known as "odd Lindley-Rayleigh Distribution". The definitions of its probability density function and cumulative distribution function using the odd Lindley-G family of distributions are provided. Some properties of the new distribution are also derived and studied in this article with applications and discussions. The estimation of the unknown parameters of the proposed distribution is also carried out using the method of maximum likelihood. The performance of the proposed probability distribution is compared to some other generalizations of the Rayleigh distribution using three simulated datasets and a real life dataset. The results obtained are compared using the values of some information criteria evaluated with the parameter estimates of the fitted distributions based on the four datasets and it is revealed that the proposed distribution outperforms all the other fitted distributions. This performance has shown that the odd Lindley-G family of distribution is an adequate generator of probability models and that the odd Linley-Rayleigh distribution is a very flexible distribution for fitting different kinds of datasets better than the other generalizations of the Rayleigh distribution considered in this study.

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1 Introduction

The Rayleigh distribution was obtained from the amplitude of sound resulting from many important sources by Rayleigh [1]. It is a continuous probability distribution with a wide range of applications such as in life testing experiments, reliability analysis, applied statistics and clinical studies. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. Its origin and other important features can be found in the work of Siddiqui [2], Hirano [3] as well as Howlader and Hossian [4].

A random variable X is said to have follow Rayleigh distribution with parameter θ if its probability density function (pdf) is given by:

$$g(x) = \theta x \, e^{-\frac{\theta}{2}x^2}$$

(1.1)

And its corresponding cumulative distribution function (cdf) is given as

$$G(x) = 1 - e^{-\frac{\theta}{2}x^2}$$

(1.2)

for $x > 0, \theta > 0$ where Θ is the scale parameter.

Due to the wide range of applications associated with the Rayleigh distribution, many authors have constructed different extensions of the distribution which have led to some flexible and good distributions such as the generalized Rayleigh distribution by Kundu et al. [5], Bivariate generalized Rayleigh distribution by Abdel-Hady [6], Transmuted Rayleigh distribution by Merovci [7], generalized Weibull-Rayleigh distribution by Yahaya and Alaku [8], Weibull-Rayleigh distribution by Merovci and Elbatal [9], transmuted Weibull-Rayleigh distribution by Yahaya and Ieren [10] as well as the Transmuted Inverse Rayleigh distribution studied by Ahmad et al. [11].

Besides these generalized Rayleigh distributions, some researchers have proven that most extended or compound distributions are more flexible and perform better than their standard counterparts when applied to real life datasets. For instance, the Weibull-Exponential distribution was found to perform better than the Exponential distribution (Oguntunde et al. [12]), the Weibull-Frechet distribution exhibited a very higher level of flexibility when applied to real life data compared to the standard Frechet distribution (Afify et al. [13]), the Lomax-Exponential distribution was also discovered to have perform better when compared to the exponential distribution during real life data analysis (Ieren and Kuhe, [14]), others are the Weibull-Lindley distribution by Ieren et al. [15], the Gompertz-Lindley distribution by Koleoso et al. [16], the Lomax-inverse Lindley distribution by Ieren et al. [17], the transmuted Lindley-Exponential distribution by Umar et al. [18], the Power Gompertz distribution by Ieren et al. [19] and many others.

Motivated by these results and evidence, it is therefore hopeful that the proposed distribution will be a more robust distribution with greater degree of skewness and flexibility. To this fact, this study aim at extending the Rayleigh distribution by using the Odd Lindley-G family to introduce a new distribution called "Odd Lindley-Rayleigh distribution".

The main aspects of this article are written in readable sections as follows. The definition of the new distribution and its plots are provided in section 2 under 2.1. Sub-section 2.2 derived some Mathematical and Statistical properties of the new distribution including estimation of unknown parameters of the proposed model using maximum likelihood estimation (MLE) provided in sub-section 2.2.7. In section 3 the proposed

distribution with some competing distributions are applied to three simulated datasets and a real life dataset under results and discussions. Finally, a brief summary with some useful conclusions are given in section 4.

2 Materials and Methods

2.1 Odd Lindley Rayleigh Distribution, OLRD

The Lindley distribution introduced by Lindley [20] in the context of Bayesian analysis as a counter example of fiducial statistics, is defined by its probability density function (pdf) and cumulative distribution function (cdf) as:

$$G(t) = 1 - \left[1 + \frac{\alpha t}{\alpha + 1} \right] e^{-\alpha t}$$

(2.1)

and

$$g(t) = \frac{\alpha^2}{\alpha + 1} (1 + t) e^{-\alpha t}$$

(2.2)

respectively. For $t > 0, \alpha > 0$, where α is the scale parameter of the Lindley distribution.

For any continuous distribution with cdf $G(x; \xi) = G(x)$ and *pdf*, $g(x; \xi) = g(x)$ Gomes-Silva et al. [21] proposed the Odd Lindley-G family of distributions that generates distributions with greater flexibility in modeling of real life datasets.

The cumulative distribution function (cdf) of the Odd Lindley-G family of distributions according to Gomes-Silva et al. [21] is defined as:

$$F(x; \alpha, \xi) = \int_{-\infty}^{\frac{G(x;\xi)}{\overline{G}(x;\xi)}} \frac{\alpha^2}{\alpha + 1} (1 + t) e^{-\alpha t} dt$$

(2.3)

where $G(x; \xi)$ is the cdf of any continuous distribution to be extended which depends on the parameter vector ξ and $\alpha > 0$ is the shape parameter of the Odd Lindley-G family while $\overline{G}(x; \xi) = 1 - G(x; \xi)$.

Using integration by substitution in the equation above and evaluating the integrand in equation (2.3) yields

$$F(x; \alpha, \xi) = 1 - \frac{\alpha + \overline{G}(x; \xi)}{(1 + \alpha) \overline{G}(x; \xi)} \exp \left\{ -\alpha \left[\frac{G(x; \xi)}{\overline{G}(x; \xi)} \right] \right\}$$

(2.4)

Therefore, equation (2.4) is the cumulative distribution function (cdf) of the Odd Lindley-G family of distributions proposed by Gomes-Silva et al. [21] and the corresponding pdf of the Odd Lindley-G family can be obtained from equation (2.4) by taking the derivative of the cdf with respect to *x* and is obtained as:

$$f(x;\alpha,\xi) = \frac{\alpha^2 g(x;\xi)}{(1+\alpha)(\bar{G}(x;\xi))^3} \exp\left\{-\alpha\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]\right\}$$

(2.5)

where $g(x;\xi)$ and $G(x;\xi)$ are the pdf and the cdf of any continuous distribution to be modified respectively which depends on the parameter vector ξ and $\alpha > 0$ is the shape parameter of the odd Lindley-G family of distributions.

Substituting equation (1.1) and (1.2) in equation (2.4) and (2.5) and simplifying, the cdf and pdf of the Odd Lindley-Rayleigh distribution are obtained as:

$$F(x) = 1 - \frac{\alpha e^{\frac{\theta}{2}x^2} + 1}{(\alpha+1)} e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)}$$

(2.6)

and

$$f(x) = \frac{\alpha^2 \theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)} \exp\left\{-\alpha\left[e^{\frac{\theta}{2}x^2}-1\right]\right\}$$

(2.7)

respectively. Hence equation (2.6) and (2.7) are the cdf and pdf of the Odd Lindley-Rayleigh distribution.

The following is a graphical representation of the pdf and cdf of the odd Lindley-Rayleigh distribution for selected parameter values.
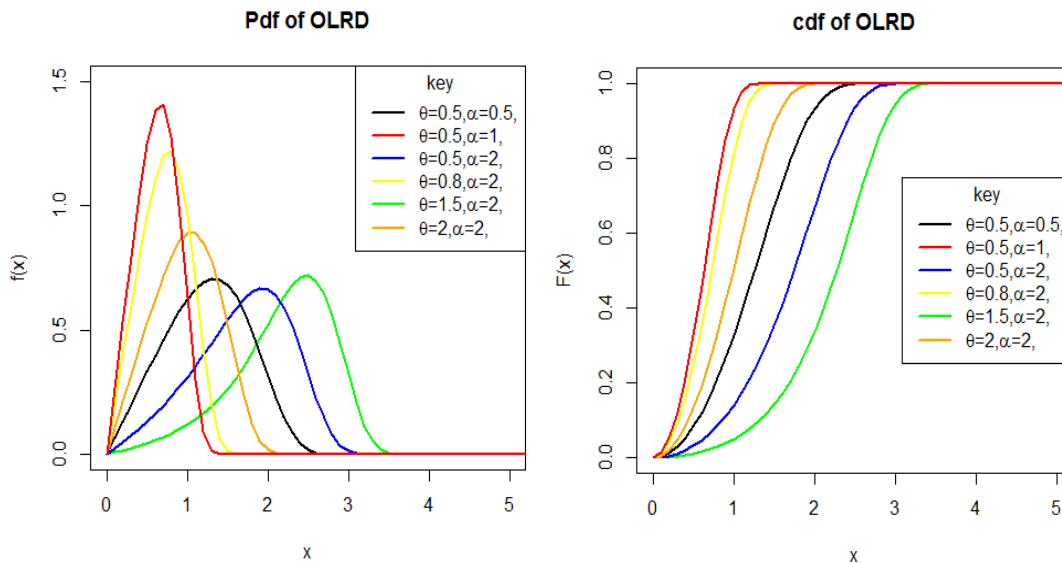

Fig. 2.1. *PDF* **and CDF of the** *OLRD* **for selected values of** θ **and** α

The plot for the pdf reveals that the *OLRD* is skewed with various shapes and therefore will be a good model for different kinds of datasets.

2.2 Mathematical and statistical properties of OLRD

In this sub-section, some properties of the *OLRD* distribution are defined and discussed vividly as follows:

2.2.1 Moments

Let X denote a continuous random variable, the nth moment of X is given by;

$$\mu_n' = E\left(X^n\right) = \int_0^\infty x^n f(x) dx$$

(2.8)

where *f(x)* the *pdf* of the odd Lindley-Rayleigh distribution is as given in equation (2.7) as:

$$f(x) = \frac{\alpha^2 \theta x e^{2\frac{\theta}{2}x^2}}{(1+\alpha)} \exp\left\{-\alpha\left[e^{\frac{\theta}{2}x^2} - 1\right]\right\}$$

(2.9)

Before substitution in (2.8), we perform the expansion and simplification of the *pdf* as follows:

First, by expanding the exponential term in (2.9) using power series, we obtain:

$$\exp\left\{-\alpha\left[e^{\frac{\theta}{2}x^2} - 1\right]\right\} = \sum_{i=0}^\infty \frac{(-1)^i \alpha^i}{i!}\left(e^{\frac{\theta}{2}x^2} - 1\right)^i$$

(2.10)

Making use of the result in (2.10) above, equation (2.9) becomes

$$f(x) = \sum_{i=0}^\infty \frac{\alpha^{2+i}\theta}{i!(1+\alpha)} x e^{2\frac{\theta}{2}x^2}\left(e^{\frac{\theta}{2}x^2} - 1\right)^i$$

(2.11)

Also, using the generalized binomial theorem, we can write the last term from the above result as:

$$\left(e^{\frac{\theta}{2}x^2} - 1\right)^i = \sum_{j=0}^\infty (-1)^j \binom{i}{j} e^{\frac{\theta}{2}(i-j)x^2}$$

(2.12)

Making use of the result in (2.12) above, equation (2.11) becomes

$$f(x) = \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j} \alpha^{2+i}\theta}{i!(1+\alpha)} \binom{i}{j} x e^{\frac{\theta}{2}(2+i-j)x^2}$$

(2.13)

Hence,

$$\mu_n' = E\left(X^n\right) = \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j} \alpha^{2+i}\theta}{i!(1+\alpha)} \binom{i}{j} \int_0^\infty x^{n+1} e^{-\frac{\theta}{2}(j-2-i)x^2} dx$$

(2.14)

Using integration by substitution in (2.14) above and simplifying, we obtain

$$\mu_n' = E\left(X^n\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}\left(\frac{2}{\theta(j-2-i)}\right)^{\frac{n}{2}+1}\int_{0}^{\infty}y^{\frac{n}{2}+1-1}e^{-y}dy$$

(2.15)

Recall that

$$\int_{0}^{\infty}y^{\frac{n}{2}}e^{-y}dy = \int_{0}^{\infty}y^{\frac{n}{2}+1-1}e^{-y}dy = \Gamma\left(\frac{n}{2}+1\right) \qquad \Gamma(\alpha) = \int_{0}^{\infty}y^{\alpha-1}e^{-y}dy$$

because

Hence,

$$\mu_n' = E\left(X^n\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}\left(\frac{2}{\theta(j-2-i)}\right)^{\frac{n+2}{2}}\Gamma\left(\frac{n+2}{2}\right)$$

(2.16)

The Mean:

The mean of the *OLRD* can be obtained from the nth moment of the distribution when $n=1$ as follows:

$$\mu_1' = E(X) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}\left(\frac{2}{\theta(j-2-i)}\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)$$

(2.17)

Also the second moment of the *OLRD* is obtained from the nth moment of the distribution when $n=2$ as

$$\mu_2' = E\left(X^2\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}\left(\frac{2}{\theta(j-2-i)}\right)^{2}$$

(2.18)

The Variance:

The nth central moment or moment about the mean of X, say μ_n, can be obtained as:

$$\mu_n = E\left(X-\mu_1'\right)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\mu_1'^i\mu_{n-i}'$$

(2.19)

The variance of X for *OLRD* is obtained from the central moment when $n=2$, that is,

$$Var(X) = E\left(X^2\right) - \left\{E(X)\right\}^2$$

(2.20)

$$Var(X) = \left(\frac{2}{\theta(j-2-i)}\right)^2\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j} - \left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^{2+i}\theta}{i!(1+\alpha)}\binom{i}{j}\Gamma\left(\frac{3}{2}\right)\right\}^2\right\}$$

(2.21)

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

2.2.2 Moment generating function

This is a more organized approach of presenting all the moments into one mathematical function, and that function is called the moment generating function (mgf). In other words, the mgf generates the moments of X by differentiation i.e., for any real number say k, the kth derivative of $M_X(t)$ evaluated at $t = 0$ is the kth moment $\mu_k^{'}$ of X.

The *mgf* of a random variable X can be obtained by:

$$M_x(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} f(x)dx$$

(2.22)

Recall that by power series expansion,

$$e^{tx} = \sum_{k=0}^\infty \frac{(tx)^k}{k!} = \sum_{k=0}^\infty \frac{t^k}{k!} x^k$$

(2.23)

Using the result in equation (2.23) and simplifying the integral in (2.22), therefore we have;

$$M_x(t) = \sum_{k=0}^\infty \frac{t^k}{k!} \left\{ \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j} \alpha^{2+i} \theta}{i!(1+\alpha)} \binom{i}{j} \left(\frac{2}{\theta(j-2-i)} \right)^{\frac{k+2}{2}} \Gamma\left(\frac{k+2}{2} \right) \right\}$$

(2.24)

2.2.3 Characteristics function

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.

The characteristics function of a random variable X is given by;

$$\varphi_x(t) = E\left(e^{itx} \right) = E\left[\cos(tx) + i\sin(tx) \right] = E\left[\cos(tx) \right] + E\left[i\sin(tx) \right]$$

(2.25)

Recall from power series expansion that

$$\cos(tx) = \sum_{k=0}^\infty \frac{(-1)^k t^{2k}}{(2k)!} x^{2k} \quad \text{and} \quad E\left[\cos(tx)\right] = \sum_{k=0}^\infty \frac{(-1)^k t^{2k}}{(2k)!} \mu_{2k}^{'}$$

And also that

$$\sin(tx) = \sum_{k=0}^\infty \frac{(-1)^k t^{2k+1}}{(2k+1)!} x^{2k+1} \quad \text{and} \quad E\left[\sin(tx)\right] = \sum_{k=0}^\infty \frac{(-1)^k t^{2k+1}}{(2k+1)!} \mu_{2k+1}^{'}$$

Simple algebra and power series expansion proves that

$$\phi_x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \mu_{2k}^{'} + i \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \mu_{2k+1}^{'}$$

(2.26)

Where μ_{2k}' and μ_{2k+1}' are the moments of X for $n=2k$ and $n=2k+1$ respectively and can be obtained from μ_k' as

$$\mu_{2n}^{'} = E\left(X^{2n}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \alpha^{2+i} \theta}{i!(1+\alpha)} \binom{i}{j} \left(\frac{2}{\theta(j-2-i)}\right)^{n+1} \Gamma(n+1)$$

and

$$\mu_{2n+1}^{'} = E\left(X^{2n+1}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \alpha^{2+i} \theta}{i!(1+\alpha)} \binom{i}{j} \left(\frac{2}{\theta(j-2-i)}\right)^{\frac{2n+3}{2}} \Gamma\left(\frac{2n+3}{2}\right)$$

respectively.

2.2.4 Reliability analysis

Survival Function: Survival function is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x)$$

(2.27)

Substituting for $F(x)$, the cdf of the odd Lindley-Rayleigh distribution and simplifying give the survival function of the proposed distribution as:

$$S(x) = \frac{\alpha e^{\frac{\theta}{2} x^2} + 1}{(\alpha+1)} e^{-\alpha\left(e^{\frac{\theta}{2} x^2} - 1\right)}$$

(2.28)

Hazard Function: Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)}$$

(2.29)

Again replacing f(x) and F(x), the pdf and cdf of the proposed odd Lindley-Rayleigh distribution and simplifying the results gives the hazard function of OLRD as:

$$h(x) = \frac{\alpha^2 \theta x e^{2\frac{\theta}{2}x^2}}{\alpha e^{\frac{\theta}{2}x^2} + 1}$$

$$(2.30)$$

The following are some possible curves for the survival function and hazard rate for selected parameters values of the model parameters.
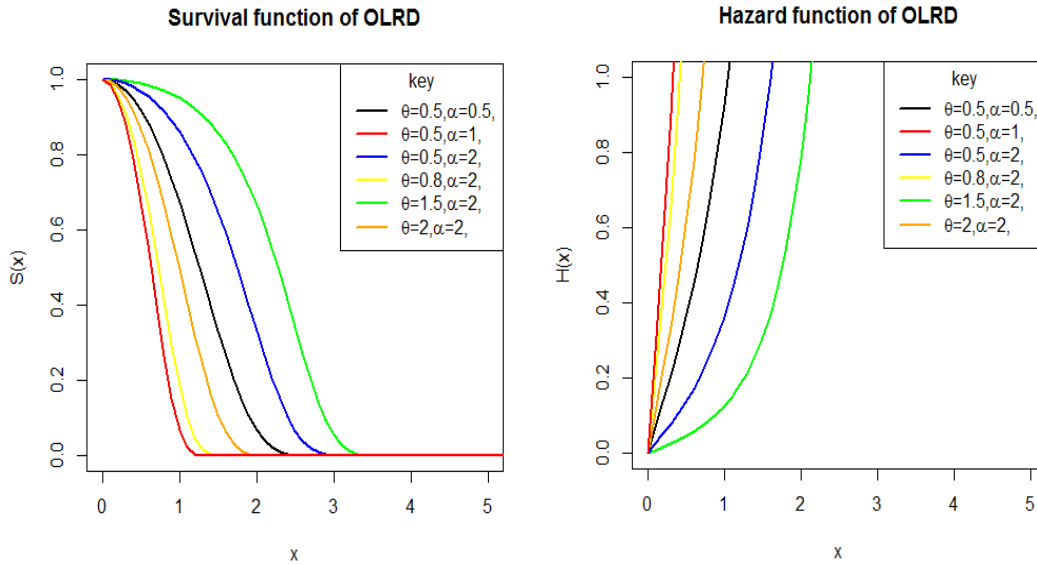

Fig. 2.2. Survival function of the *OLRD* at different parameter values as shown on the plot using the key above

The graphs in Fig. 2.2 show that the probability of survival equals one (1) at initial time or early age and it decreases as X (time) increases and equals zero (0) as X approaches infinity. It can also be seen that the hazard function increases as X (time) increases. This means that the *OLRD* may be appropriate for modeling time dependent events, where risk or hazard increases with time or age.

2.2.5 Quantile function

Hyndman and Fan [22] defined the quantile function for any distribution in the form $Q(u) = X_q = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$.

Taking *F(x)* to be the *cdf* of the OLRD and inverting it as above will give us the quantile function as follows:

$$F(x) = 1 - \frac{\alpha + e^{-\frac{\theta}{2}x^2}}{(1+\alpha)e^{-\frac{\theta}{2}x^2}} \exp\left\{-\alpha\left[\frac{1-e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}}\right]\right\} = u$$

$$(2.31)$$

Solving equation (2.31) above gives:

$$-(\alpha+1)(1-u)e^{-(\alpha+1)} = -\frac{\alpha + e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}} \exp\left\{-\frac{\alpha + e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}}\right\}$$

$$(2.32)$$

Using the expression in (2.32) above, it can be seen that $-\dfrac{\alpha+e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}}$ is the Lambert function of the real

argument, $-(\alpha+1)(1-u)e^{-(\alpha+1)}$ since the Lambert function is defined as: $w(x)e^{w(x)}=x$.

Also note that the Lambert function has two branches with a branching point located at $\left(-e^{-1},1\right)$. The lower branch, $W_{-1}(x)$ is defined in the interval $\left[-e^{-1},1\right]$ and has a negative singularity for $x\to 0^{-1}$. The upper branch, $W_0(x)$, is defined for $x\in\left[-e^{-1},\infty\right]$. Hence, equation (2.32) can be written as:

$$W\left(-(\alpha+1)(1-u)e^{-(\alpha+1)}\right)=-\frac{\alpha+e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}}$$

(2.33)

Now for any $\alpha>0$ and $u\in(0,1)$, it follows that $\dfrac{\alpha+e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}}>1$ and $\left((\alpha+1)(1-u)e^{-(\alpha+1)}\right)<0$.

Therefore, considering the lower branch of the Lambert function, equation (2.33) can be presented as:

$$W_{-1}\left(-(\alpha+1)(1-u)e^{-(\alpha+1)}\right)=-\frac{\alpha+e^{-\frac{\theta}{2}x^2}}{e^{-\frac{\theta}{2}x^2}}$$

(2.34)

Collecting like terms in equation (2.34) and simplifying the result, the quantile function of the OLRD is obtained as:

$$Q(u)=\sqrt{\frac{2}{\theta}\log\left\{-\frac{1}{\alpha}\left[W_{-1}\left(-(\alpha+1)(1-u)e^{-(\alpha+1)}\right)+1\right]\right\}}$$

(2.35)

where u is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function.

The median of *X* from the OLRD is simply obtained by setting *u*=0.5 and this substitution of $u=0.5$ in equation (2.35) gives:

$$MD=\sqrt{\frac{2}{\theta}\log\left\{-\frac{1}{\alpha}\left[W_{-1}\left(-\tfrac{1}{2}(\alpha+1)e^{-(\alpha+1)}\right)+1\right]\right\}}$$

(2.36)

Similarly, random numbers can be simulated from the OLRD by setting $Q(u)=X$ and this process is called simulation using inverse transformation method. This means for any $\alpha,\theta>0$ and $u\in(0,1)$:

$$X=\sqrt{\frac{2}{\theta}\log\left\{-\frac{1}{\alpha}\left[W_{-1}\left(-(\alpha+1)(1-u)e^{-(\alpha+1)}\right)+1\right]\right\}}$$

(2.37)

"where u is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function".

Again using the quantile function above, the quantile based measures of skewness and kurtosis are obtained as follows:

Kennedy and Keeping [23] defined the Bowley's measure of skewness based on quartiles as:

$$ SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} $$

(2.38)

And Moors [24] presented the Moors' kurtosis based on octiles by:

$$ KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} $$

(2.39)

"where $Q(.)$ is calculated by using the quantile function from equation (2.35).

2.2.6 Order statistics

Sample values such as the smallest, largest, or middle observation from a random sample provide important information. For example, the highest rainfall, flood or minimum temperature recorded during past years might be useful when planning for future emergencies. Let $X_{(1)}$ denote the smallest of $X_1, X_2, ..., X_n$, $X_{(2)}$ denote the second smallest of $X_1, X_2, ..., X_n$, and similarly $X_{(i)}$ denote the ith smallest of $X_1, X_2, ..., X_n$. Then the random sample, $X_{(1)}, X_{(2)}, ..., X_{(n)}$, called the order statistics of the sample $X_1, X_2, ..., X_n$, thus the pdf of the ith order statistic, $X_{(i)}$, is given by:

$$ f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} $$

(2.40)

Where $f(x)$ and $F(x)$ are the *pdf* and *cdf* of the proposed distribution respectively.

Using (2.6) and (2.7), the *pdf* of the ith order statistics $X_{i:n}$, can be expressed from (2.40) as

$$ f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\alpha^2 \theta x e^{2\frac{\theta}{2}x^2} e^{-\alpha\left[e^{\frac{\theta}{2}x^2}-1\right]}}{(1+\alpha)} \right] \left[1 - \frac{\alpha e^{\frac{\theta}{2}x^2}+1}{(\alpha+1)} e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)} \right]^{i+k-1} $$

(2.41)

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *OLRD* are given by;

$$f_{1:n}(x) = n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha^2\theta x e^{\frac{\theta}{2}x^2} e^{-\alpha\left[e^{\frac{\theta}{2}x^2}-1\right]}}{(1+\alpha)}\right]\left[1-\frac{\alpha e^{\frac{\theta}{2}x^2}+1}{(\alpha+1)}e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)}\right]^k$$

(2.42)

and

$$f_{n:n}(x) = n\left[\frac{\alpha^2\theta x e^{\frac{\theta}{2}x^2} e^{-\alpha\left[e^{\frac{\theta}{2}x^2}-1\right]}}{(1+\alpha)}\right]\left[1-\frac{\alpha e^{\frac{\theta}{2}x^2}+1}{(\alpha+1)}e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)}\right]^{n-1}$$

(2.43)

respectively.

2.2.7 Estimation of parameters of OLRD using maximum likelihood method

Let $X_1, X_2,...,X_n$ be a sample of size "n" independently and identically distributed random variables from the *OLRD* with unknown parameters α and θ defined previously.

The likelihood function is given by:

$$L\left(\underline{X}\mid\alpha,\theta\right) = \frac{\left(\alpha^2\theta\right)^n\prod_{i=1}^n x_i}{(1+\alpha)^n}e^{\frac{2\theta}{2}\sum_{i=1}^n x^2}e^{-\alpha\sum_{i=1}^n\left(e^{\frac{\theta}{2}x^2}-1\right)}$$

(2.44)

Let the log-likelihood function be $l = \log L\left(\underline{X}\mid\alpha,\theta\right)$, therefore

$$l = 2n\log\alpha + n\log\theta - n\log(1+\alpha) + \sum_{i=1}^n\log x_i + \theta\sum_{i=1}^n x_i^2 - \alpha\sum_{i=1}^n\left[e^{\frac{\theta}{2}x^2}-1\right]$$

(2.45)

Differentiating l partially with respect to α and θ respectively gives:

$$\frac{\partial l}{\partial\alpha} = \frac{2n}{\alpha} - \frac{n}{(\alpha+1)} - \sum_{i=1}^n\left(e^{\frac{\theta}{2}x_i^2}-1\right)$$

(2.46)

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^n x_i^2 - \frac{\alpha}{2}\sum_{i=1}^n\left\{\frac{\left(x_i^2 e^{\frac{\theta}{2}x^2}\right)}{\left(e^{\frac{\theta}{2}x^2}-1\right)}\right\}$$

(2.47)

The solution of the non-linear system of equations of $\dfrac{dl}{d\alpha} = 0$, and $\dfrac{\partial l}{\partial \theta} = 0$ will give us the maximum

likelihood estimates of parameters α and θ. However, the solution cannot be gotten analytically except numerically with the aid of suitable statistical software like R, SAS, e.t.c when data sets are available. Hence, the package named "AdequacyModel" was used in R software to estimate the parameters of the proposed model during the process of applications.

3 Results and Discussion

This section presents three simulated and one real life datasets, their descriptive statistics, graphical summary and applications using some selected generalizations of the Rayleigh distribution to the datasets. Five models (the Transmuted Weibull-Rayleigh distribution (*TWRD*), Weibull-Rayleigh distribution (*WRD*), the Transmuted Rayleigh distribution (*TRD*), Lindley distribution (*LD*) and the Rayleigh distribution (*RD*)) are applied to the four datasets together with the proposed distribution (OLRD) and their performance is being evaluated and compared under this section using four information criteria.

In order to evaluate and compare the performance of the models listed above, some model selection criteria have been utilized which include *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *HQIC* (Hannan Quin information criterion). The formulas for these statistics are given as follows:

$$AIC = -2ll + 2k \text{ , } BIC = -2ll + k\log(n), \ CAIC = -2ll + \frac{2kn}{(n-k-1)} \text{ and}$$
$$HQIC = -2ll + 2k\log\big[\log(n)\big]$$

where ll denotes the log-likelihood value evaluated with the maximum likelihood estimates (MLEs), k is the number of model parameters and **n** is the sample size. The model with the lowest values of these statistics would be chosen as the best model to fit any of the datasets.

Data set I: This data set represents 25 identically and independently distributed random samples from the proposed distribution using its quantile function at some selected values of the parameters. Its summary is given as follows:

Table 3.1. Descriptive statistics for dataset I

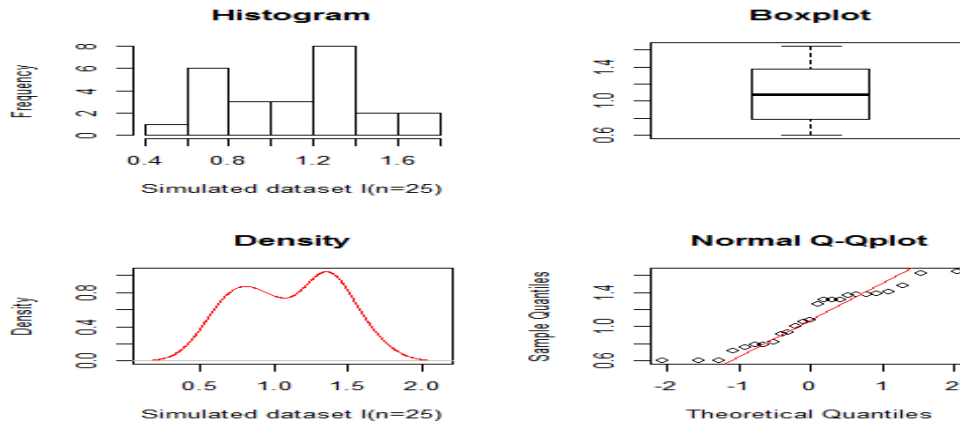| n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
25	0.2394	0.832	1.1166	1.5173	1.1187	1.8964	0.2025	-0.1833	-0.9056

Fig. 3.1. A graphical summary of Dataset I

From the descriptive statistics in Table 3.1 and the histogram, box plot, density and normal Q-Q plot shown in Fig. 3.1 above, we observed that dataset (dataset I) is approximately normal, that is, neither skewed to the right nor left and therefore not suitable for distributions that are skewed.

Table 3.2. Performance of the distribution using the *AIC*, *CAIC*, *BIC* and *HQIC* values of the models based on simulated dataset I when n=25

| Distributions	Parameter estimates	-ll	AIC	CAIC	BIC	HQIC	Ranks of models
OLRD	$\hat{\theta}$ =1.1199 $\hat{\alpha}$ =1.1414	10.0423	24.0845	24.6300	26.5223	24.7607	1
TWRD	$\hat{\theta}$ =0.1260 $\hat{\alpha}$ =9.5775 $\hat{\beta}$ =0.8422 $\hat{\lambda}$ =-0.4783	21.9526	51.9052	53.9052	56.7807	53.2575	5
WRD	$\hat{\theta}$ =0.2779 $\hat{\alpha}$ =8.5750 $\hat{\beta}$ =1.3773	32.5378	71.0755	72.2184	74.7322	72.0897	6
TRD	$\hat{\theta}$ =0.7230 $\hat{\lambda}$ =0.9985	14.7251	33.4502	33.9957	35.8880	34.1263	3
RD	$\hat{\theta}$ =1.5325	14.6776	31.3551	31.5290	32.5740	31.6932	2
LD	$\hat{\alpha}$ =1.3262	24.9092	51.8183	51.9923	53.0372	52.1564	4
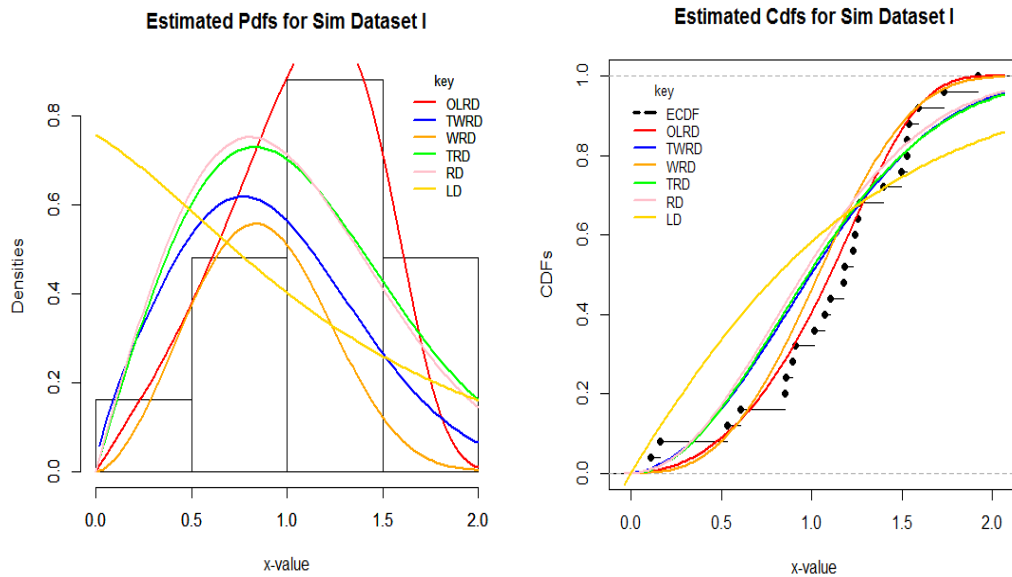
Fig. 3.2. Histogram and plots of the estimated densities (pdfs) and cdfs of the OLRD, TWRD, WRD, TRD, RD and LD fitted to dataset I

Table 3.2 presents the parameter estimates and the values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* for the six fitted models using simulated data when the sample size is 25 (n=25). The values in the above table reveal that the *OLRD* has better performance with the lowest values of *AIC*, *CAIC*, *BIC* and *HQIC* compared to the other five models.

The Fig. 3.2 displayed the histogram and estimated densities and cdfs of the fitted models for dataset I.
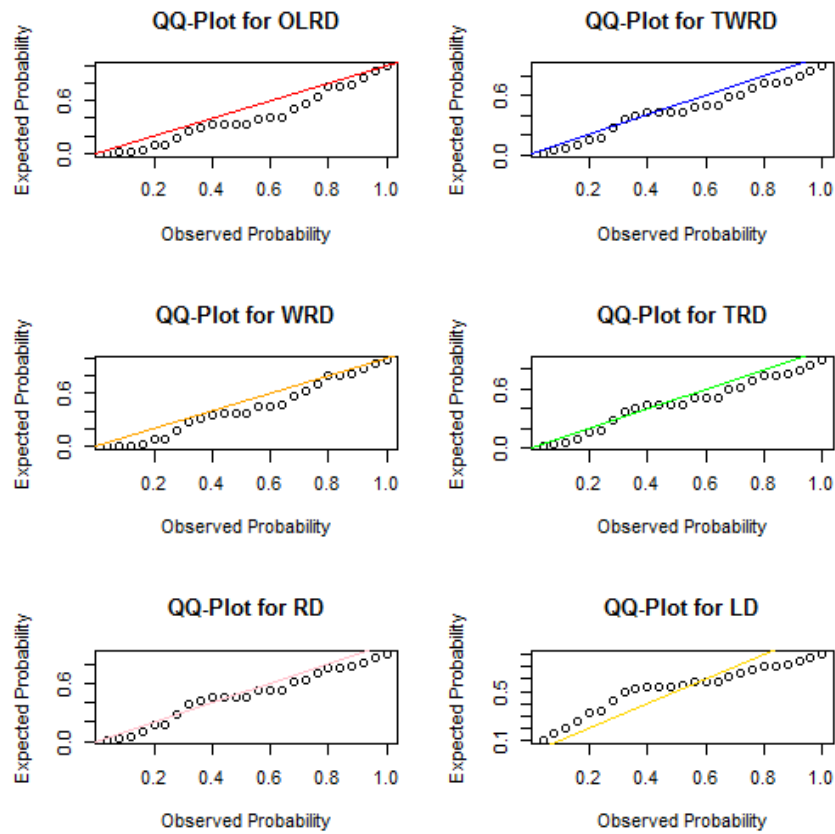
Fig. 3.3. Probability plots for the fit of the OLRD, TWRD, WRD, TRD, RD and LD based on dataset I

From the estimated density plots in Fig. 3.2 it is very clear that the OLRD fits the data much better than the other five distributions. This performance shows that the OLRD is flexible and can take different shapes or fit various datasets because the first dataset (dataset I) is normally distributed while the OLRD is skewed and can be more useful for skewed datasets. Again the Q-Q plots also confirm that fact that even though the OLRD has a better fit for the data, it is not a very good model because the dataset used is normally distribution while the distribution is a skewed distribution.

Data set II: This data set represents 75 random samples from the odd Lindley-Rayleigh distribution obtained by using the quantile function derived from the distribution. These sample values are summarized as follows:

Table 3.3. Descriptive statistics for dataset II

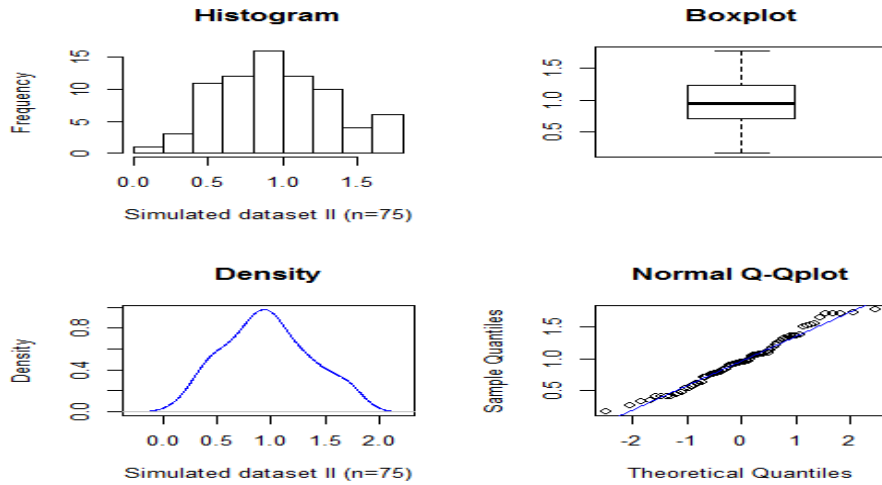| n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
75	0.0514	0.6751	1.0105	1.3029	0.9702	1.9777	0.2108	-0.1463	-0.6230

Fig. 3.4. A graphical summary of Dataset II

Based on the descriptive statistics from Table 3.3 and the histogram, box plot, density and normal Q-Q plot from Fig. 3.4 we can see that dataset (dataset II) is normal, that is, not skewed and therefore not suitable for distributions that are skewed.

Table 3.4. Performance of the distribution using the *AIC, CAIC, BIC and HQIC* values of the models based on simulated dataset II when n=75

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models
OLRD	$\hat{\theta}$ =1.0090 $\hat{\alpha}$ =1.3602	33.7660	71.5319	71.6986	76.1669	73.3826	1
TWRD	$\hat{\theta}$ =0.1535 $\hat{\alpha}$ =9.6230 $\hat{\beta}$ =0.9222 $\hat{\lambda}$ =-0.3811	67.3211	142.642	143.2137	151.9122	146.3436	5
WRD	$\hat{\theta}$ =0.2779 $\hat{\alpha}$ =8.5750 $\hat{\beta}$ =1.3773	96.4854	198.971	199.3087	205.9232	201.7468	6
TRD	$\hat{\theta}$ =0.8426 $\hat{\lambda}$ =0.9452	43.5922	91.1843	91.3509	95.8192	93.0350	3
RD	$\hat{\theta}$ =1.6791	43.0834	88.1667	88.2215	90.4842	89.0921	2
LD	$\hat{\alpha}$ =1.3968	70.7841	143.5681	143.6229	145.8856	144.4935	4

Again, Table 3.4 also gives the parameter estimates and the values of all the performance measures for the six fitted distributions when the sample is increased to 75 (n=75). Based on the values in the above table, we

also conclude that the *OLRD* fits the simulated data better than the other five distributions (*TWRD, TRD, RD,WRD & LD*) and is the most fitted for this dataset (n=75).

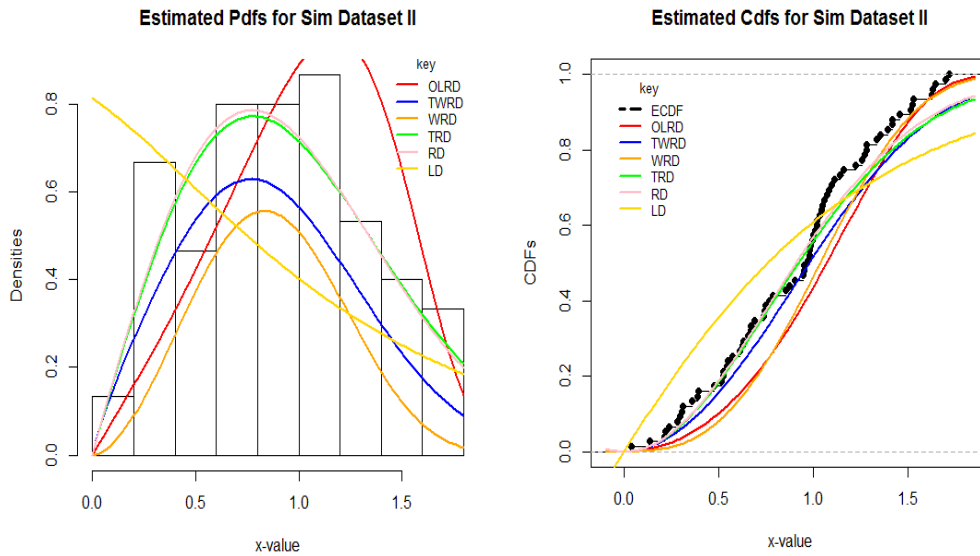The following figure displayed the histogram and estimated densities and cdfs of the fitted models for dataset II.

Fig. 3.5. Histogram and plots of the estimated densities (pdfs) and cdfs of the OLRD, TWRD, WRD, TRD, RD and LD fitted to dataset II
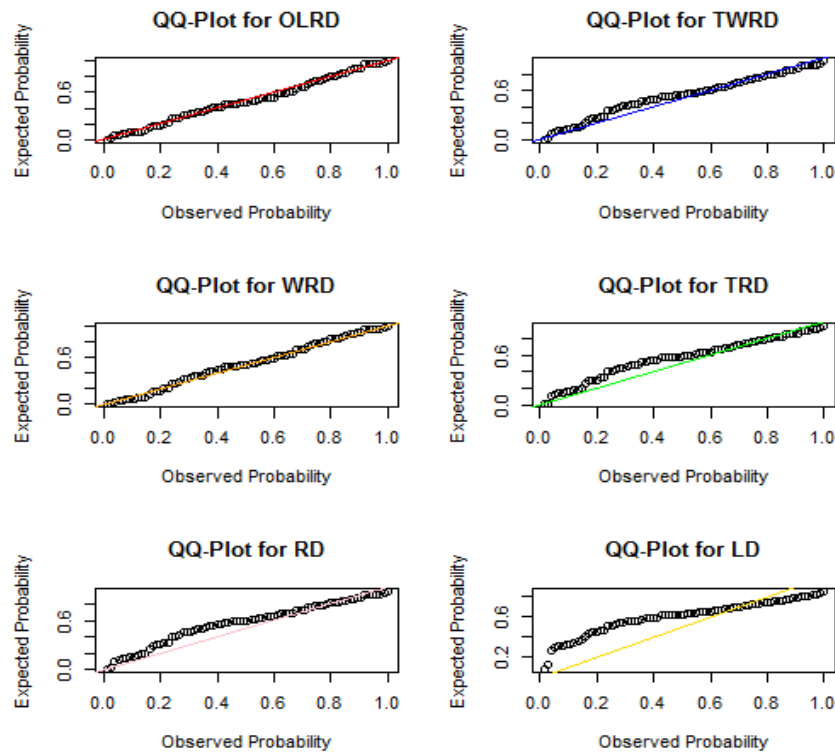
Fig. 3.6. Probability plots for the fit of the OLRD, TWRD, WRD, TRD, RD and LD based on dataset II

Using the estimated densities in Fig. 3.5 and the Q-Q plots in Fig. 3.6, it is again shown that the OLRD though not a normal model has a better fit to the data than the other distributions (TWRD, WRD, TRD, RD and LD) which again confirm the fact that it is a very flexible distribution and can take the form of various datasets irrespective their distribution or nature.

Data set III: The third dataset represents 125 samples simulated from the odd Lindley-Rayleigh distribution using the quantile function of the distribution. The descriptive statistics for this data are as follows:

Table 3.5. Descriptive statistics for data set III

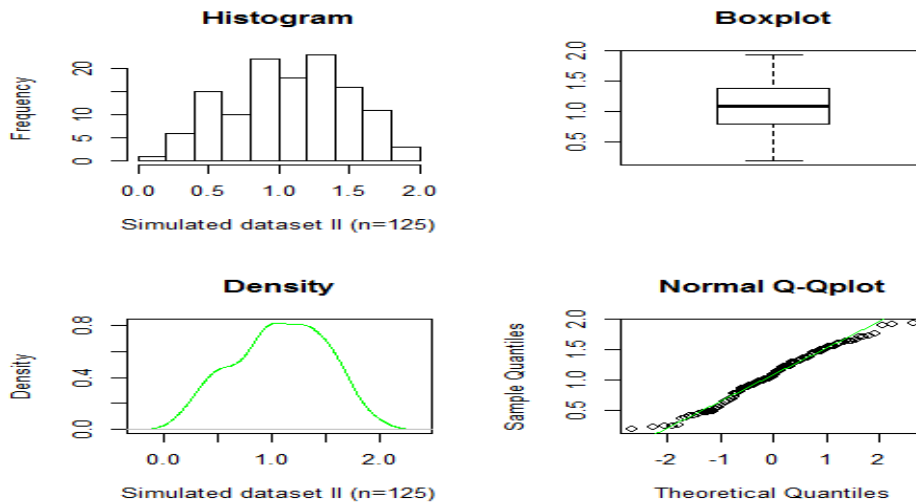| n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
125	0.1150	0.6933	0.9586	1.236	0.957	1.795	0.1602	-0.0222	-0.6587

Fig. 3.7. A graphical summary of Dataset III

The descriptive statistics in Table 3.5 and the histogram, box plot, density and normal Q-Q plot shown in Fig. 3.7 above show that dataset (dataset III) is normal.

Table 3.6. Performance of the distribution using the *AIC*, *CAIC*, *BIC* and *HQIC* values of the models based on simulated dataset III when n=125

Distributions	Parameter estimates	-ll=(-log-likelihood value)	AIC	CAIC	BIC	HQIC	Ranks of models
OLRD	$\hat{\theta}$ =0.7375 $\hat{\alpha}$ =2.2749	61.7909	127.5819	127.6802	133.2385	129.8799	1
TWRD	$\hat{\theta}$ =0.0529 $\hat{\alpha}$ =5.9760 $\hat{\beta}$ =0.7987	105.9874	219.9748	220.3081	231.2881	224.5708	4

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models
	$\hat{\lambda}$ =0.8809						
WRD	$\hat{\theta}$ =0.2779	156.3396	318.6792	318.8775	327.1641	322.1262	6
	$\hat{\alpha}$ =8.5750						
	$\hat{\beta}$ =1.3773						
TRD	$\hat{\theta}$ =0.8904	70.04084	144.0817	144.1800	149.7383	146.3797	3
	$\hat{\lambda}$ =0.9853						
RD	$\hat{\theta}$ =1.82017	69.7792	141.5584	141.5909	144.3867	142.7074	2
LD	$\hat{\alpha}$ =1.4483	113.1006	228.2012	228.2337	231.0295	229.3502	5

We can as well see from this table, Table 3.6 that the proposed distribution, *OLRD* has smaller values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* compared to the other five distributions.

The following figure displayed the histogram and estimated densities and cdfs of the fitted models for dataset III.
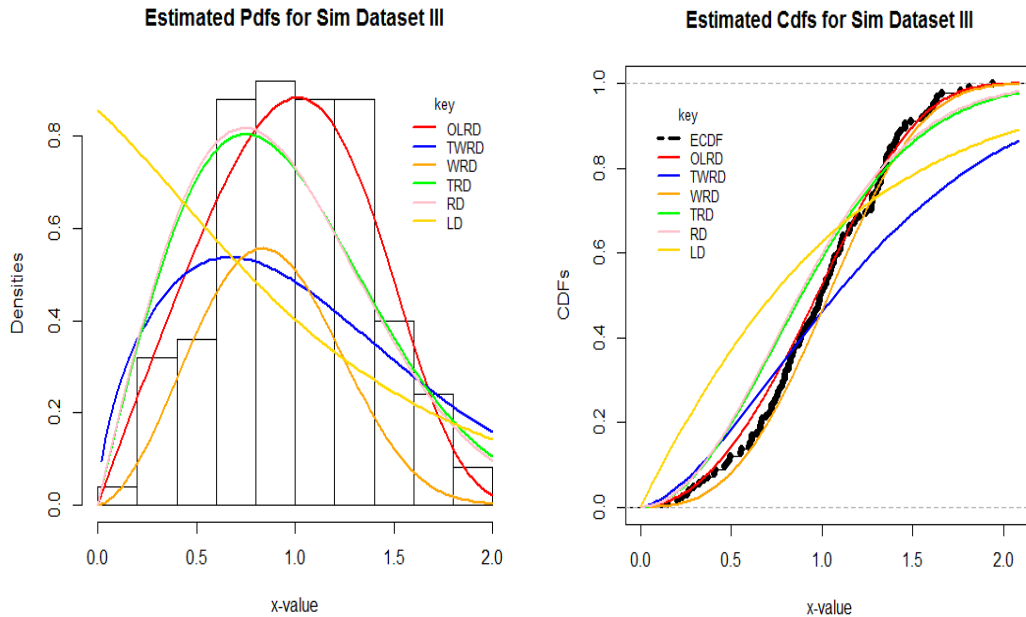

Fig. 3.8. Histogram and plots of the estimated densities (pdfs) and cdfs of the OLRD, TWRD, WRD, TRD, RD and LD fitted to dataset III

Similarly, the estimated densities in Fig. 3.8 and the Q-Q plots in Fig. 3.9, show that the OLRD though not a normal distribution has a better fit to the simulated data for n=125 (dataset III) than the other five distributions (TWRD, WRD, TRD, RD and LD) which proves the fact that it is more flexible than the other distributions and can model various datasets no matter the distribution of the data.
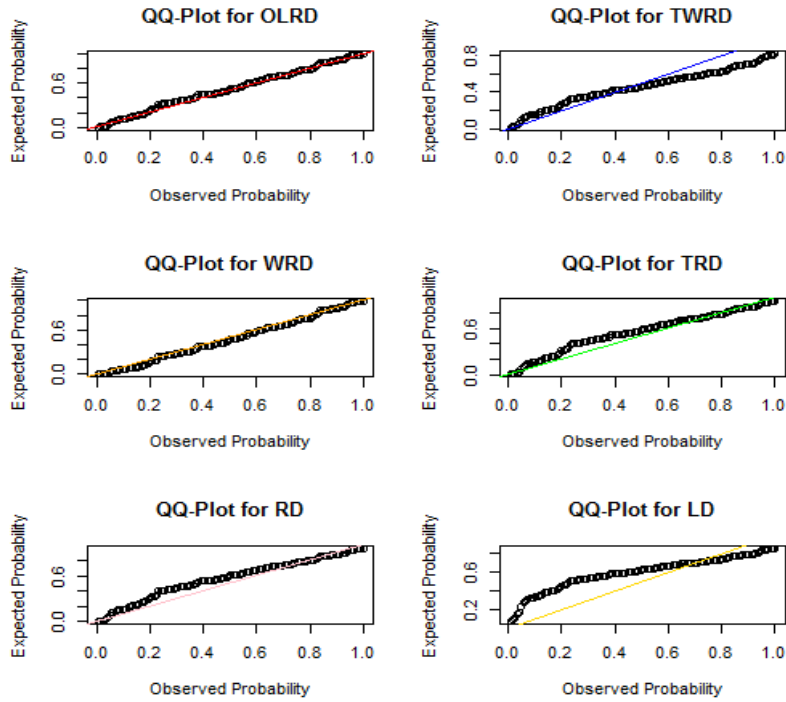
Fig. 3.9. Probability plots for the fit of the OLRD, TWRD, WRD, TRD, RD and LD based on dataset III

Data set IV: This is a real life dataset and it represents the strength of 1.5 cm glass fibers initially collected by members of staff at the UK national laboratory. It has been used by Afify and Aryal [25], Barreto-Souza et al. [26], Bourguignon et al. [27], Oguntunde et al. [12], Ieren et al. [15], Ieren and Yahaya [28], Yahaya and Ieren [29], Koleoso et al. [16] as well as Smith and Naylor [30]. Its summary is given as follows:

Table 3.7. Descriptive Statistics for the real life dataset (Dataset IV)

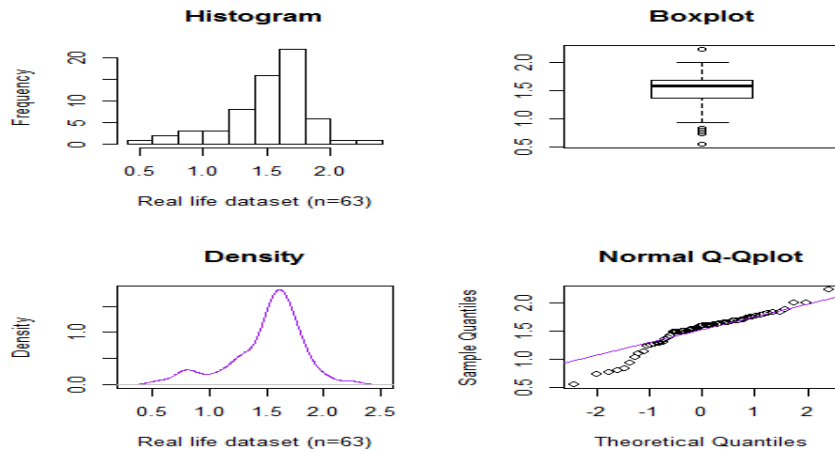| n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

Fig. 3.10. A graphical summary the real life data (Dataset IV)

Considering the descriptive statistics in Table 3.7 and the histogram, box plot, density and normal Q-Q plot shown in Fig. 3.10 above, it is revealed that the real life dataset (dataset IV) is negatively skewed, that is, skewed to the left and therefore most suitable for distributions that are skewed just like the OLRD.

Table 3.8. Performances of the distributions using the *AIC, CAIC, BIC and HQIC* values of the models based on the real life dataset (the strength of 1.5 cm glass fibers)

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models
OLRD	$\hat{\theta}$=1.5255 $\hat{\alpha}$=0.2648	15.1622	34.3244	34.5244	38.6107	36.0102	1
TWRD	$\hat{\theta}$=0.0551 $\hat{\alpha}$=8.0918 $\hat{\beta}$=0.7582 $\hat{\lambda}$=-0.7960	67.0918	142.1835	142.8732	150.7561	145.5552	4
WRD	$\hat{\theta}$=0.1708 $\hat{\alpha}$=7.7896 $\hat{\beta}$=1.5159	90.9095	187.8191	188.2258	194.2485	190.3478	6
TRD	$\hat{\theta}$=0.7840 $\hat{\lambda}$=0.0169	50.1225	104.245	104.445	108.5313	105.9308	3
RD	$\hat{\theta}$=0.8425	49.7909	101.5818	101.6474	103.7249	102.4247	2
LD	$\hat{\alpha}$=0.9962	81.2785	164.5569	164.6225	166.7000	165.3998	5

Again we can as well see from the table above that the *OLRD* has smaller values of -*ll*, *AIC*, *BIC*, *CAIC* and *HQIC* compared to the other five distributions using the real life dataset. The values in the above table also provide evidence for us to agree that the *OLRD* fits both the real life and simulated data better than the other five models. This therefore proves that the *OLRD* could be used to model all kinds of data sets both real and simulated.

The Fig. 3.11 displayed the histogram and estimated densities and cdfs of the fitted models for dataset IV.

Looking at the estimated densities (pdfs) in Fig. 3.11 it can be seen clearly that the OLRD has a better fit to the real life dataset (dataset IV) than TWRD, WRD, TRD, RD and LD. The reason is that the real life dataset (dataset IV) is skewed to the left while the OLRD is also a skewed model and hence should perform better the other five distributions. Again the Q-Q plots in Fig. 3.12 also confirm that the proposed distribution is more flexible than the other five distributions as already shown previously with simulated datasets of sizes n=25, n=75 and n=125.
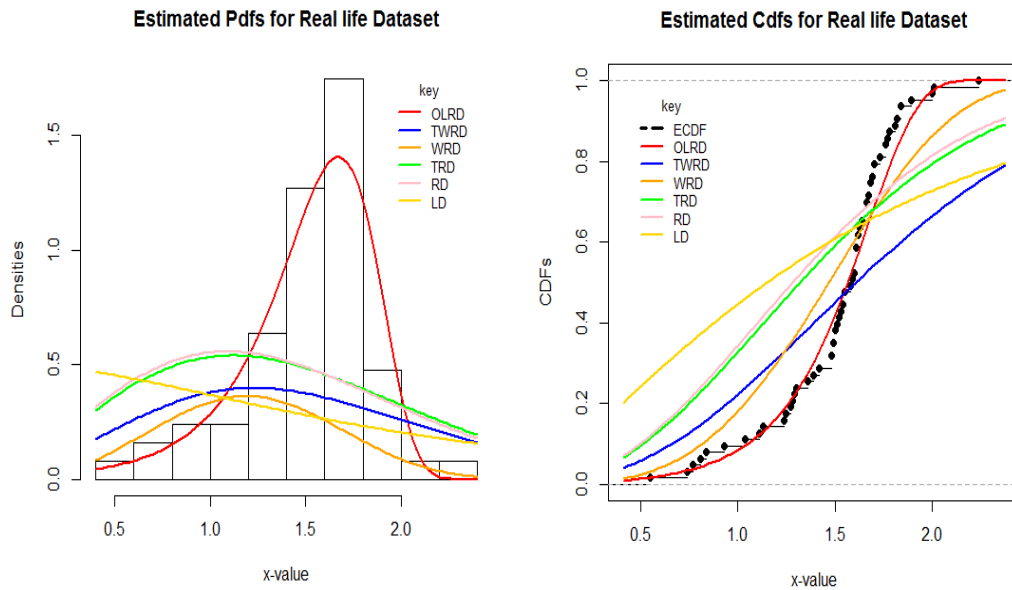
Fig. 3.11. Histogram and plots of the estimated densities (pdfs) and cdfs of the OLRD, TWRD, WRD, TRD, RD and LD fitted to dataset IV
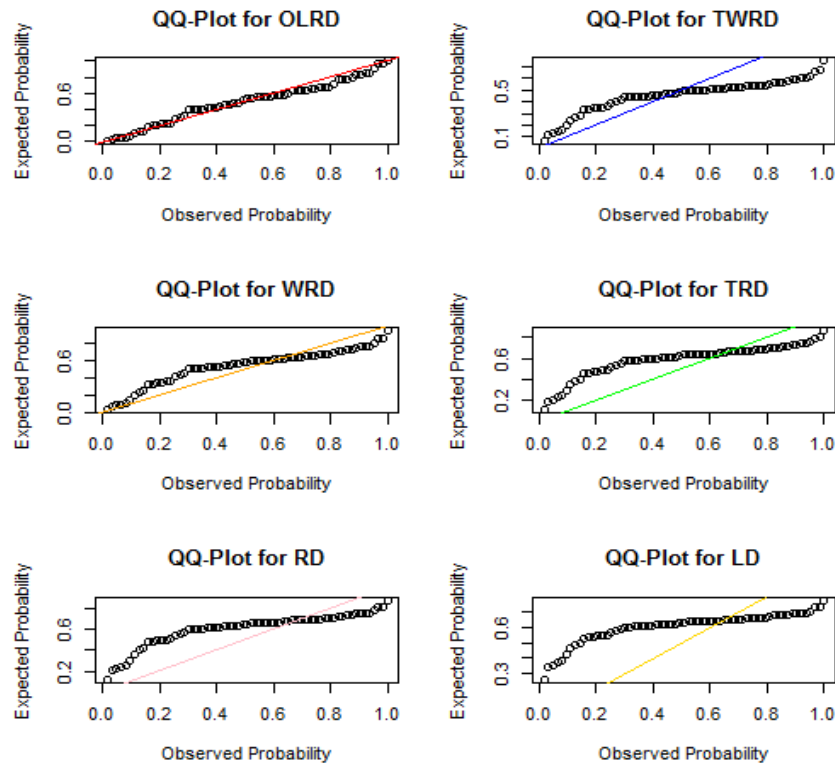

Fig. 3.12. Probability plots for the fit of the OLRD, TWRD, WRD, TRD, RD and LD based on dataset IV

4 Conclusion

In this paper, a new two-parameter continuous distribution has been proposed named "odd Lindley-Rayleigh distribution". Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its validity, moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done appropriately. Some plots of the distribution revealed that it can take any shape depending on values of the parameters. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the *OLRD* has a decreasing survival rate and an increasing failure function. The results of the three applications to simulated datasets and the one real life dataset showed that the proposed distribution (odd Lindley-Rayleigh distribution) performs better than the transmuted Weibull-Rayleigh distribution (*TWRD*), Weibull-Rayleigh distribution (*WRD*), transmuted Rayleigh distribution (*TRD*), Lindley distribution (*LD*) and the Rayleigh distribution (*RD*) irrespective of the nature of the data sets and the sample sizes. This implies that the *OLRD* is a very flexible model and can be used for all forms of data in different fields. Based on the usefulness and performance of this distribution (OLRD) and the significance of parameter estimation methods in the application of any model, it is recommended that future research in this area should compare different methods of estimation on the two parameters of the proposed probability distribution (Odd Lindley-Rayleigh distribution).

Competing Interests

Authors have declared that no competing interests exist.

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