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# Weyl's Theorem for Algebraically $(\wp, \wr, \varrho)$-Paranormal and Algebraically $(\wp, \wr, \varrho)$-*-Paranormal Operators

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

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## Abstract

Present $L$ be an algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators on $L^2$ space. We examine Weyl's theorem, $\mathfrak{a}$-Browder's theorem and spectral mapping theorem holds for weyl's spectrum of $L$ and essential approximate point spectrum of $L$.

*Keywords: Weyl's theorem; $\mathfrak{a}$-Browder's theorem; algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators.*

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# 1 Introduction

Let $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{K}(\mathfrak{H})$ stand for the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space, respectively, throughout this paragraph. If $L$ is in the range of $\mathfrak{B}(\mathfrak{H})$, then we write $N(L)$ for $L$'s null space and $R(L)$ for its range. Furthermore, let $\sigma(L)$, $\sigma_a(L)$ and $\pi_0(L)$ stand for the spectrum, approximation point spectra, and point spectrum of $L$, respectively, and let $\alpha(L) = dimN(L)$, $\beta(L) = dimN(L)$ [1, 2, 3, 4]. If an operator $L \in \mathfrak{B}(\mathfrak{H})$ has a closed range, a finite dimensional null space, and a finite co-dimension in its range, it is referred to as a Fredholm operator. A Fredholm operator's index can be found using the formula

$$i(L) = \alpha(L)\beta(L)$$

$L$ is referred to as Browder if it is a Fredholm with finite ascent and descent, and Weyl if it is a Fredholm of index zero[5]. Comparatively, if $L$ is Fredholm and $L - ⅄$ is invertible for small enough $\mid ⅄ \mid > 0$, then $⅄ \in C$. Based on [6, 7, 8, 9, 10], we may construct the essential spectrum $\sigma_b(L)$, the Weyl spectrum $\omega(L)$, and the Browder spectrum $\sigma_b(L)$ of $L$.

$$\sigma_e(L) = \{⅄ \in C : L - ⅄ \ is\ not\ Fredholm\},$$

$$\omega(L) = \left\{⅄ \in C : L - ⅄ \ is\ not\ Weyl's\right\},$$

and

$$\sigma_b(L) = \{⅄ \in C : L - ⅄ \ is\ not\ Browder\},$$

Based on the evidence [11, 12],

$$\sigma_e(L) \subseteq \omega(L) \subseteq \sigma_b(L) = \sigma_e(L) \cup acc\sigma(L),$$

where $acc\mathfrak{K}$ represents the accumulation points of $\mathfrak{K} \subseteq C$. Write $\mathfrak{K}\ acc\mathfrak{K}$, and we let

$$\pi_{00}(L) = \{⅄ \in iso\sigma(L) : 0 < \sigma(L - ⅄) < \infty\},$$

and

$$p_{00}(L) = \sigma(L)\ \sigma_b(L).$$

Weyl's theorem holds for algebraically paranormal operators, as demonstrated recently by Raul E. Curto and Young Min Han [13], D. Senthilkumar, and P. Mageshwari naik[14]. We examine several characteristics of absolute-(p, r)-*-paranormal operators after demonstrating that Weyl's theorem holds for algebraically absolute-$(p, r)$-*-paranormal operators and M.H.M. Rashid [15]. This result is extended to algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators in this note.

# 2 Weyl's Theorem for Algebraically $(\wp, \wr, \varrho)$-Paranormal Operator

An operator $L$ is $(\wp, \wr, \varrho)$-paranormal for each $\wp > 0$, $\wr \geq 0$ and $\varrho > 0$ if, for each unit vector $\Im \in \mathfrak{H}$, $\left\| |L|^{\wp} \mho |L|^{\wr} \Im \right\|^{\frac{1}{\varrho}} \geq \left\| |L|^{\frac{\wp+\wr}{\varrho}} \Im \right\|$.

If there is a nonconstant complex polynomial $p$ such that $p(L)$ is $(\wp, \wr, \varrho)$-Paranormal then we now prove a result for the algebraically $(\wp, \wr, \varrho)$-paranormal operator.

**Lemma 2.1.** *Assume that $\sigma(L) = \{⅄\}$ and let $L$ be a $(\wp, \wr, \varrho)$-paranormal operator with $⅄ \in C$. Consequently, $L = ⅄$.*

*Proof.* We examine two scenarios: Case I ($⅄ = 0$): $L$ is normaloid because it is $(\wp, \wr, \varrho)$-paranormal. $L$ thus equals 0.

In case II ($\curlywedge \neq 0$), we observe that $L^{-1}$ is also $(\wp, \wr, \varrho)$-paranormal because $L$ is invertible and $L$ is $(\wp, \wr, \varrho)$-paranormal. $L^{-1}$ is hence normaloid.

Conversely, since $\sigma(L^{-1}) = \frac{1}{\curlywedge}$, so $\|L\| \left\|L^{-1}\right\| = |\curlywedge| \left|\frac{1}{\curlywedge}\right| = 1$. Given that $L$ is convexoid, as deduced from [Mla, Lemma 3], $W(L) = \curlywedge$. $T$ thus equals $\curlywedge$. ☐

**Lemma 2.2.** *Consider an algebraically $(\wp, \wr, \varrho)$-paranormal operator $L$. Then $L$ is nilpotent.*

*Proof.* Let $p$ be a nonconstant polynomial $p$ such that $p(L)$ is $(\wp, \wr, \varrho)$-paranormal. $p(L)-p(0)$ is a quasinilpotent operator since $\sigma(p(L)) = p(\sigma(L))$. Assuming $m \geq 1$, it follows from Lemma 2.1 that $cL^m(L-\curlywedge_1)...(L-\curlywedge_n) \equiv p(L) - p(0) = 0$. For any $\curlywedge_i \neq 0$, $L - \curlywedge_i$ is invertible, so $L^m = 0$. ☐

**Lemma 2.3.** *Let $L$ be a algebraically $(\wp, \wr, \varrho)$-paranormal operator. Then $L$ is isoloid.*

*Proof.* Assume that $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - L)^{-1}d\mu$ is the associated Riesz idempotent and that $\curlywedge \in iso\sigma(L)$. $D$ is a closed disk with $\curlywedge$ at its center and no additional points of $\sigma(L)$. The direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \; where \; \sigma(L_1) = \{\curlywedge\} \; and \; \sigma(L_2) = \sigma(L) \backslash \{\curlywedge\}$$

For some nonconstant polynomial $p$, $p(L)$ is $(\wp, \wr, \varrho)$-paranormal since L is algebraically $(\wp, \wr, \varrho)$-paranormal. We must have $\sigma(p(L_1)) = p(\sigma(L_1)) = \{p(\sigma)\}$. since $p(L_1) - p(\curlywedge)$ is hence quasipotent. Lemma 2.1 states that since $p(L_1)$ is $(\wp, \wr, \varrho)$-paranormal, $p(L_1) - p(\curlywedge) = 0$. Assign $q(z)$ to $p(z) - p(\curlywedge)$.As a result, $L_1$ is algebraically $(\wp, \wr, \varrho)$-paranormal and $q(L_1) = 0$. Lemma 2.2 indicates that $L_1 - \curlywedge$ is nilpotent as it is quasinilpotent and algebraically $(\wp, \wr, \varrho)$-paranormal. As a result, $\curlywedge \in \pi_0(L_1)$ and $\curlywedge \in \pi_0(L)$. This demonstrates $L$ is isoloid. ☐

**Theorem 2.4.** *Let $L$ is algebraically $(\wp, \wr, \varrho)$-paranormal operator. Then Weyl's theorem holds for $f(L)$ for every $f \in \mathfrak{H}(\sigma(L))$.*

*Proof.* We first prove that $L$ is a valid case of Weyl's theorem. Assume $\curlywedge$ lies within $\curlywedge \in \sigma(L) \backslash \omega(L)$. Hence, $L - \curlywedge$ is not invertible and is Weyl. As we assert, $\curlywedge \in \partial\sigma(L)$. Contrarily, suppose that $\curlywedge$ is an inner point of $\sigma(L)$. Then, for any $\mu \in U$, there exists a neighborhood $U$ of $\curlywedge$ such that $dimN(L - \mu) > 0$. It is evident from [Fin, Theorem 10] that $L$ lacks SVEP. Conversely, [[16], Corollary 2.10] implies that $p(L)$ possesses SVEP since $p(L)$ is $(\wp, \wr, \varrho)$-paranormal for some nonconstant polynomial $p$.

Therefore, $L$ possesses SVEP by [[17], Theorem 3.3.9], which is contradictory. Consequently, $\curlywedge$ lies within $\curlywedge \in \partial\sigma(L) \backslash \omega(L)$, and the punctured neighborhood theory implies that $\curlywedge \in \pi_{00}(L)$.

Conversely, consider the case where $\curlywedge \in \pi_{00}(L)$ and the corresponding Riesz idempotent $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - L)^{-1}d\mu$, where $D$ is a closed disk with $\curlywedge$ at its center and no other points of $\sigma(L)$. Then, just as previously, we can express $L$ as the direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \; where \; \sigma(L_1) = \{\curlywedge\} \; and \; \sigma(L_2) = \sigma(L) \backslash \{\curlywedge\}$$

We examine two scenarios: Case I ($\curlywedge = 0$): Since $L_1$ is quasinilpotent and algebraically $(\wp, \wr, \varrho)$-paranormal in this case, Lemma 2.2 implies that $L_1$ is nilpotent. As we say, $dimR(P) < \infty$. For, $0 \notin \pi_{00}(L)$, which would be contradictory if $N(L_1)$ were infinite dimensional. Weyl follows from the fact that $L_1$ is a finite dimensional operator. Nonetheless, we may determine that $L$ is Weyl because $L_2$ is invertible. As a result, $0 \in \sigma(L) \backslash \omega(L)$.

Case II ($\curlywedge \neq 0$): Lemma 2.3 proof shows that $L_1 - \curlywedge$ is nilpotent. $L_1 - \curlywedge$ is Weyl since $\curlywedge \in \pi_{00}(L)$ and $L_1 - \curlywedge$ is a finite dimensional operator. $L - \curlywedge$ is Weyl since $L_2 - \curlywedge$ is invertible. Weyl's theorem therefore applies to $L$. Next, we assert that for any $f \in \mathfrak{H}(\sigma(L))$, $f(\omega(L)) = \omega(f(L))$. Assume that $f \in \mathfrak{H}(\sigma(L))$. Without any more restrictions on $L$, $\omega(f(L)) \subseteq f(\omega(L))$; therefore, proving that $f(\omega(L)) \subseteq \omega(f(L))$ is sufficient. Assume that $\curlywedge \neq \omega(f(L))$. In that case, $g(L)$ is invertible,

$$f(L) - \curlywedge = c(L - \alpha_1)(L - \alpha_2)...(L - \alpha_n)g(L), \tag{2.1}$$

and $f(L) - \curlywedge$ is Weyl. Every $L - \alpha_i$ is Fredholm since the operators on the right side of (2.1) commute. $L$ has SVEP [[16], Corollary 2.10], since $L$ is algebraically $(\wp, \wr, \varrho)$-paranormal. From [[18], Theorem 2.6], we can infer that for every $i = 1, 2, ..., n$, $i(L - \alpha_i) \leq 0$. Consequently, $\curlywedge \notin f(\omega(L))$ and hence $f(\omega(L)) = \omega(f(L))$.

Now remember (Lemma) that for any $f \in \mathfrak{H}(\sigma(L))$,

$$f(\sigma(L) \backslash \pi_{00}(L)) = \sigma(f(L)) \backslash \pi_{00}(f(L))$$

This is true if $L$ is isoloid. Weyls theorem holds for $L$ since it is isoloid (Lemma 2.3) and so,

$$\sigma(f(L)) \backslash \pi_{00}(f(L)) = f(\sigma(L) \backslash \pi_{00}(L)) = f(\omega(L)) = \omega(f(L))$$

means that Weyls theorem applies for $f(L)$. The proof is now complete. □

**Corollary 2.5.** *Let L be an algebraically $(\wp, \wr, \varrho)$-paranormal operator. For each $f \in \mathfrak{H}(\sigma(L))$, we get $\omega(f(L)) = f(\omega(L))$.*

# 3 Weyl's Theorem for Algebraically $(\wp, \wr, \varrho)$-*-Paranormal Operator

An operator $L$ is $(\wp, \wr, \varrho)$-*-paranormal for each $\wp > 0$, $\wr \geq 0$ and $\varrho > 0$ if $\left\| |L|^{\wp} \mho |L|^{\wr} \Im \right\|^{\frac{1}{\varrho}} \geq \left\| |L|^{\frac{\wp+\wr}{\varrho}} \mho^* \Im \right\|$. If there is a nonconstant complex polynomial $p$ such that $p(L)$ is $(\wp, \wr, \varrho)$-*-paranormal, we say that $L$ is algebraically $(\wp, \wr, \varrho)$-*-paranormal. We now prove a result for the algebraically $(\wp, \wr, \varrho)$-*-paranormal operator.

**Lemma 3.1.** *Assume that $\sigma(L) = \{\curlywedge\}$ and let L be a $(\wp, \wr, \varrho)$-*-paranormal operator with $\curlywedge \in C$. Consequently, $L = \curlywedge$.*

**Lemma 3.2.** *Let L be a algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. Then L is nilpotent.*

**Lemma 3.3.** *Let L be a algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. Then L is isoloid.*

**Theorem 3.4.** *Let L be an algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. Then, for each $f \in H(\sigma(L))$, Weyl's theorem holds for $f(L)$.*

**Corollary 3.5.** *Let L be an algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. For each f that is in $\mathfrak{H}(\sigma(L))$, we have $\omega(f(L)) = f(\omega(L))$.*

# 4 $\mathfrak{a}$-Browder's Theorem for Algebraically $(\wp, \wr, \varrho)$-Paranormal and Algebraically $(\wp, \wr, \varrho)$-*-Paranormal Operators

Generally speaking, we cannot assume that operators with only SVEP are covered by Weyl's theorem. Take a look at this instance: Let

$$L(x_1, x_2, x_3, ...) = \left(\frac{1}{2}x_2, \frac{1}{2}x_3, ...\right)$$

define $L \in \mathfrak{B}(l_2)$. Since $L$ is quasinilpotent, $L$ possesses SVEP. However, Weyl's theorem does not apply to $L$ since $\sigma(L) = \omega(L) = \{0\}$ and $\pi_{00}(L) = \{0\}$. Nevertheless, as Theorem 3.4 below demonstrates, $\mathfrak{a}$-Browder's theory holds for $L$. First, we require the following auxiliary result, which is mostly the work of C.K. Fong [Fon]; we offer a proof for completeness. Remember that if $\mathfrak{X} \in \mathfrak{B}(\mathfrak{H})$ has a dense range and a trivial kernel, it is referred to be a quasiaffinity. If there is a quasiaffinity $\mathfrak{X}$ such that $\mathfrak{X}\mathfrak{S} = L\mathfrak{X}$, then $\mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ is considered a quasiaffine transform of $L$. If $\mathfrak{S}$ and $L$ both $\mathfrak{S} \prec L$ and $L \prec \mathfrak{S}$, then we say that $\mathfrak{S}$ and $L$ are quasisimilar.

**Lemma 4.1.** *let $\mathfrak{S} \prec L$ and assume that $L$ possesses SVEP. Next, $\mathfrak{S}$ possesses SVEP.*

**Theorem 4.2.** *Assume that $L$ or $L^*$ is algebraically $(\wp, \wr, \varrho)$-paranormal. For each $f \in \mathfrak{H}(\sigma(L))$, we get $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$.*

**Theorem 4.3.** *Assume that $L$ or $L^*$ is algebraically $(\wp, \wr, \varrho)$-*-paranormal. For each $f \in \mathfrak{H}(\sigma(L))$, we get $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$.*

**Theorem 4.4.** *Let $\mathfrak{S} \prec L$ and assume that $L$ possesses SVEP. For each $f \in \mathfrak{H}(\sigma(L))$, then $\mathfrak{a}$-Browder's theorem holds for $f(\mathfrak{S})$.*

**Corollary 4.5.** *Assume that $\mathfrak{S} \prec L$ and that $L$ is an algebraically $(\wp, \wr, \varrho)$-paranormal operator. For each $f \in \mathfrak{H}(\sigma(L))$, then $\mathfrak{a}$-Browder's theorem holds for $f(\mathfrak{S})$.*

**Corollary 4.6.** *Assume that $\mathfrak{S} \prec L$ and that $L$ is an algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. For each $f \in \mathfrak{H}(\sigma(L))$, then $\mathfrak{a}$-Browder's theorem holds for $f(\mathfrak{S})$.*

# 5 Conclusion

The study establishes the validity of Weyl's theorem, a-Browder's theorem, and the spectral mapping theorem in the context of algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators. Our examination reveals that these theorems hold for the Weyl's spectrum and the essential approximate point spectrum of the operator $L$, thereby contributing to a deeper understanding of the spectral properties of such operators.

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## Competing Interests

Authors have declared that no competing interests exist.

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