



## Modeling of Gauss Elimination Technique and AHA Simplex Algorithm for Multi-objective Linear Programming Problems

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### Authors' contributions

This work was carried out in collaboration between both authors. Author SJ designed the study, performed the modeling of Gauss elimination technique for LPP, wrote the elimination procedure and wrote the first draft of the manuscript. Author AM managed the analyses of the study to reduce MOLPP into LPP by AHA algorithm and then solved by Gauss elimination technique. Both the authors read and approved the final manuscript.

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## Abstract

In this research paper, an effort has been made to solve each linear objective function involved in the Multi-objective Linear Programming Problem (MOLPP) under consideration by AHA simplex algorithm and then the MOLPP is converted into a single LPP by using various techniques and then the solution of LPP thus formed is recovered by Gauss elimination technique. MOLPP is concerned with the linear programming problems of maximizing or minimizing, the linear objective function having more than one objective along with subject to a set of constraints having linear inequalities in nature. Modeling of Gauss elimination technique of inequalities is derived for numerical solution of linear programming problem by using concept of bounds. The method is quite useful because the calculations involved are simple as compared to other existing methods and takes least time. The same has been illustrated by a numerical example for each technique discussed here.

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## 1 Introduction

Mathematical modeling is an important tool to describe the different characteristics of an observed phenomenon (a problem) and their interactivity as well as their dynamics through Mathematics in optimization. It provides us a way to simplify observed phenomenon and to reach at its computationally description. Mathematical modeling is a simplification of real world problems. This is an art of constructing problems from an application area into manageable mathematical formulations whose numerical and theoretical analysis provides guidance, perception and answers which are useful for generating other applications.

This is an interesting, innovative, interdisciplinary, growing and important subject for researchers since many interesting algorithm have been introduced by researchers and still have lots of possibilities to find more new algorithms which must be easy, appropriate and less time taking. Some algorithms have been computationally tested. Each algorithm or method has some merits and demerits. In the framework of each application, some algorithm or method seems more suitable than others.

Sen, C. [1] developed a new approach for MOLPP. According to his suggestion, a multi-objective function should be constructed under the limitation in such a way that the optimum value of individual problem is greater than zero. The solution of linear multi-objective programming problems had been carried out by Sulaiman, N. A. and Sadiq, G. W. [2] by using mean and median value. An optimal transformation technique to find the solution of MOLPP had been developed by Sulaiman, N. A. and O. Hamadameen, A. Q. [3]. Hamad-Amin A. O. [4] discussed an adaptive arithmetic average technique to solve MOLPP. Sulaiman, N. A. and Mustafa, R. B. [5] proposed harmonic mean technique to solve multi-objective linear programming problems. Nahar, S. and Alim, M. A. [6] proposed a new statistical averaging method to solve MOLPP. Different elimination techniques for linear programming problems have been studied earlier by Kohler [7], Williams [8], Kanniappan et al. [9], Sharma et al. [10]. Jain et al. [11,12] proposed Gauss as well as Modified Fourier elimination techniques for linear fractional programming problems. Further, Jain et al. [13,14] presented extended Gauss as well as extended Modified Fourier elimination technique for integer solution of linear fractional programming problem. Jain, S. [15] proposed the technique to find the solution of multi-objective linear programming problem by modeling of Gauss elimination technique. Jain, S. [16] proposed modeling of Fourier elimination technique for multi-objective fractional programming problem. Jain, S. [17] presented the modeling of Gauss elimination technique for multi-objective fractional programming problems; in which, he considered all the linear objective functions as constraints to solve MOLPP. Here, we have taken a multi-objective linear programming problem and applied the AHA simplex algorithm to solve linear programming problem for multi-objective functions. After that, the single LPP has been constructed by using various techniques (Chandra Sen, averaging techniques and new averaging techniques) from the MOLPP under consideration and then solved it by Gauss elimination technique. We solved the LPP thus formed by all these techniques one by one by Gauss elimination technique and the result thus obtained are compared with other existing techniques available in the literature and found the same answer. The optimal answer found by this technique is the same as the result obtained from either simplex method or by graphical method (in case of only two decision variables) or by AHA simplex algorithm. The AHA simplex algorithm had been introduced to solve Linear Programming Problem (LPP) by Ansari, A. H. [18]. Jain et al. [19] commented on the result obtained by Nahar, S. and Alim, M. A. [6] in the case of new harmonic average technique.

In the later section, description about methodology is presented followed by AHA simplex algorithm and Gauss elimination technique for inequalities. Further, problem formulation is described and after that various techniques to convert MOLPP into a single LPP has been discussed.

## 2 Methodology

The mathematical programming problem can be solved by various techniques available in the literature so far but two new techniques namely Gauss elimination method and AHA simplex algorithm are discussed here in details:

### 2.1 AHA simplex algorithm

Step 1: We have to construct as many separate linear programming problems as the number of objectives involved in the multi-objective linear programming problem under consideration.

Step 2: To apply AHA simplex algorithm, firstly we have to write each linear programming problem in the form given below:

$$\text{Maximize } Z_i \leq c_i^T x + \alpha_i$$

or

$$\text{Maximize } [1 \quad -c^T] \begin{bmatrix} Z_i \\ x \end{bmatrix} \leq 0$$

$$\text{Subject to, } [0 \quad A] \begin{bmatrix} Z_i \\ x \end{bmatrix} \leq b$$

$$\text{and } x \geq 0$$

Step 3: The objective function of every linear program must be in maximization form. If it is not so, then we have to convert it into maximize form by using the standard result i.e., Maximize  $Z_i = -$  Minimize  $(Z_i)$

Step 4: The requirement vector components i.e.,  $b_i$  must be positive always. If any one of  $b_i$  is not positive, then we have to make it positive by multiplying the corresponding inequality by -1.

Step 5: Now, we have to check the sign convention of all the decision variables involved in the objective inequality.

- (i) The solution will be an optimal one if all  $-c_j \geq 0$ ; where  $Z = 0$  and  $x_j = 0, \forall j$
- (ii) There is a requirement to improve the solution if at least one  $-c_j < 0$ , then go to the next step.

Step 6: Select the most negative value involved among all  $-c_j$ . Let it be for  $j = k$ .

- (i) If all  $a_{ik}$  are negative for all  $i$  then there exists an unbounded solution of the given linear programming problem.
- (ii) If there exists at least one  $a_{ik}$  is positive then the vector  $x_k$  corresponding to it enters the basis. The column in which  $x_k$  lies is known as pivotal column.
- (iii) Select the minimum ratio amongst the ratio  $\left\{ \frac{b_i}{a_{ik}}; \forall a_{ik} > 0, \forall i \right\}$ . Let minimum ratio occurs for  $\left\{ \frac{b_s}{a_{sk}}; \forall a_{sk} > 0 \right\}$ , then the row in which  $a_{sk}$  lies is said to be pivotal row; which is left blank in initial iteration. The intersection of pivotal row and pivotal column is known as pivotal element i.e.  $a_{sk}$ .

Step 7: To construct the next table for iteration, proceed similarly as the process adopted for ordinary simplex algorithm.

Step 8: Go to step 5. Repeat the procedure until either an optimal solution is achieved.

### 2.2 Gauss elimination technique for inequalities

In Gauss elimination technique, we solve the system of simultaneous equation with the help of elimination of variables one by one and finally the above system reduces to upper triangular system of equations, which can be solved by back substitution. Here, we are applying Gauss elimination technique for a system of inequalities of the same nature i.e., either less than or equal to ( $\leq$ ) or greater than or equal to ( $\geq$ ). In this technique, the variables are eliminated by combining the inequalities in such a way that the inequalities and variables reduce in one iteration so at last there remains only one variable with one inequality remains. This last inequality gives value of last variable in bounded form and finally taking the value of last variable either maximum or minimum according to the objective function of LPP reduced from MOLPP. Finally, we get value of other variables by back substitution of value of the remaining variables [12].

For the sake of clarity and simplicity, we consider the system of n variables and m inequalities:

$$a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + \dots + a_{2n}x_n \leq b_2$$

.....

.....

$$a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$$

To eliminate the first variable say  $x_1$ , multiply the first row by  $\frac{a_{21}}{a_{11}}, \frac{a_{31}}{a_{11}}, \dots, \frac{a_{m1}}{a_{11}}$  respectively and then subtract them from second, third and so on upto the last row. Then we get first iteration, which is

$$a_{22}^{(2)}x_2 + \dots + a_{2n}^{(2)}x_n \leq b_2^{(2)}$$

$$a_{32}^{(2)}x_2 + \dots + a_{3n}^{(2)}x_n \leq b_3^{(2)}$$

.....

.....

$$a_{m2}^{(2)}x_2 + \dots + a_{mn}^{(2)}x_n \leq b_m^{(2)}$$

Where,  $a_{mn}^{(2)} = a_{mn} - \frac{a_{m1}a_{1n}}{a_{11}}$  and  $b_m^{(2)} = b_m - b_1 \frac{a_{m1}}{a_{11}}$

Now after first iteration, we get (n-1) variables and (m-1) inequalities. Repeating this process or after (n-1) iterations we have only one variable remains. Finally, one can get the value of other variables by back substitution of the value of last variable. It is possible that some redundant constraints may present in the system.

### 3 Problem Formulation

Here we consider the Multi-objective linear programming problem as:

$Z = \{ Z_1, Z_2, \dots, Z_s \}$  where the objectives is to find the

Maximum value of  $Z_1, Z_2, \dots, Z_r$  &

Minimum value of  $Z_{r+1}, \dots, Z_s$ .

subject to,  $Ax \leq b$

and  $x \geq 0$

where, A is a matrix of order  $m \times n$  and  $S = \{x \in R^n | Ax \leq b, x \geq 0 \text{ and } b \in R^m (x \geq 0)\}$  is a feasible set and  $s \geq 2$ .

Also,  $Z_i = c_i^T x + \alpha_i$ ;  $c_i^T \in R^n, \alpha_i \in R \forall i = 1, 2, 3, \dots, s$

### 4 Techniques to Convert MOLPP into a Single LPP

Firstly we have to solve each LPP by simplex method or by graphical method (if no. of decision variables is two only) or by any other method and find the value of objective functions.

Let us suppose

Max. $Z_1 = \tau_1$
Max. $Z_2 = \tau_2$
.....
Max. $Z_r = \tau_r$
Min. $Z_{r+1} = \tau_{r+1}$
.....
Min. $Z_s = \tau_s$

#### 4.1 Chandra Sen’s technique

Now, as per Chandra Sen’s technique, we can convert MOLPP into a single LPP as follows:

$$\text{Max. } Z = \sum_1^r \frac{Z_i}{|\tau_i|} - \sum_{r+1}^s \frac{Z_i}{|\tau_i|} = \alpha x_1 + \beta x_2 \text{ (let)} \tag{4.1}$$

where  $\tau_i$  is non-zero for  $i = 1, 2, \dots, s$

subject to the constraints given in the problem under consideration. Here  $\tau_i$  may be positive or negative and  $\tau_i$  denotes the value of objective function of  $i^{\text{th}}$  objective.

#### 4.2 Arithmetic averaging technique

$$\text{Max } Z = \sum_1^r \frac{Z_i}{\text{A.M.}(AA_i)} - \sum_{r+1}^s \frac{Z_i}{\text{A.M.}(AL_i)} = \alpha x_1 + \beta x_2 \text{ (let)} \tag{4.2}$$

Where  $AA_i = |\tau_i|$  for  $i = 1, 2, \dots, r$ ;  $AL_i = |\tau_i|$  for  $i = r + 1, \dots, s$

A.M. here refers for Arithmetic Mean.

### 4.3 Geometric averaging technique

$$\text{Max } Z = \sum_1^r \frac{Z_i}{\text{G.M.}(AA_i)} - \sum_{r+1}^s \frac{Z_i}{\text{G.M.}(AL_i)} = \alpha x_1 + \beta x_2 \text{ (let)} \quad (4.3)$$

Where  $AA_i = |\tau_i|$  for  $i = 1, 2, \dots, r$ ;  $AL_i = |\tau_i|$  for  $i = r + 1, \dots, s$

G.M. here refers for Geometric Mean.

### 4.4 Harmonic averaging technique

$$\text{Max } Z = \sum_1^r \frac{Z_i}{\text{H.M.}(AA_i)} - \sum_{r+1}^s \frac{Z_i}{\text{H.M.}(AL_i)} = \alpha x_1 + \beta x_2 \text{ (let)} \quad (4.4)$$

Where  $AA_i = |\tau_i|$  for  $i = 1, 2, \dots, r$ ;  $AL_i = |\tau_i|$  for  $i = r + 1, \dots, s$

H.M. here refers for Harmonic Mean.

### 4.5 New averaging techniques

Let  $m_1 = \text{Minimum } \{AA_i\}$  where  $AA_i = |\tau_i|$  and  $\tau_i$  is the maximum value of  $Z_i$  for  $i = 1, 2, \dots, r$ . Also let  $m_2 = \text{Minimum } \{AL_i\}$  where  $AL_i = |\tau_i|$  and  $\tau_i$  is the minimum value of  $Z_i$  for  $i = r + 1, \dots, s$ .

#### 4.5.1 New arithmetic averaging technique

Arithmetic average will be  $m = \frac{m_1 + m_2}{2}$

$$\text{Hence, objective function becomes Max. } Z = \frac{\sum_1^r Z_i - \sum_{r+1}^s Z_i}{m} = \alpha x_1 + \beta x_2 \text{ (let)} \quad (4.5)$$

#### 4.5.2 New geometric averaging technique

Geometric average will be  $m = \sqrt{m_1 m_2}$

$$\text{Hence, objective function becomes Max. } Z = \frac{\sum_1^r Z_i - \sum_{r+1}^s Z_i}{m} = \alpha x_1 + \beta x_2 \text{ (let)} \quad (4.6)$$

#### 4.5.3 New harmonic averaging technique

Harmonic average will be  $m = \frac{2}{\frac{1}{m_1} + \frac{1}{m_2}}$

$$\text{Hence, objective function becomes Max. } Z = \frac{\sum_1^r Z_i - \sum_{r+1}^s Z_i}{m} = \alpha x_1 + \beta x_2 \text{ (let)} \quad (4.7)$$

After that, by taking objective function as constraint and all constraints of the same sign of inequality, reduced form of MOLPP for Gauss elimination technique is as follows:

$$\begin{aligned} &\text{Max } Z \\ &Z - \alpha x_1 - \beta x_2 \leq 0 \\ &A x \leq b \\ &-x \leq 0 \end{aligned}$$

Now, we have to combine the inequalities in such a way that the inequalities and the variables are reduced one by one in each iteration. If an absurd inequality  $0 \leq d$  is found at any stage, where  $d$  is a negative number then the given MOLPP has infeasible solution. Otherwise one can get the feasible solution.

### 5 Numerical Example

Solve the given multi-objective linear programming problem: (Samsun, N. et al (2017))

$$\begin{array}{ll}
 \text{Maximize} & Z_1 = x_1 + 2 x_2 \\
 \text{Maximize} & Z_2 = x_1 + 0 x_2 \\
 \text{Minimize} & Z_3 = - 2x_1 - 3 x_2 \\
 \text{Minimize} & Z_4 = 0 x_1 - x_2 \\
 \text{Subject to} & 6x_1 + 8 x_2 \leq 48 \\
 & x_1 + x_2 \geq 3 \\
 & x_1 + 0 x_2 \leq 4 \\
 & 0x_1 + x_2 \leq 3 \\
 \text{and} & x_1, x_2 \geq 0
 \end{array}$$

First of all, we have to solve each and every linear programming problem along with the given constraints to find the optimal value of each objective of MOLPP by any of the methods available in the literature. Here, we have applied AHA Simplex Algorithm to find the optimal value of each objective function. For the first linear program, the procedure is as under:

$$\begin{array}{ll}
 \text{(I)} \quad \text{Max.} & Z_1 = x_1 + 2 x_2 \\
 \text{s.t.} & 6x_1 + 8 x_2 \leq 48 \\
 & x_1 + x_2 \geq 3 \\
 & x_1 + 0 x_2 \leq 4 \\
 & 0x_1 + x_2 \leq 3 \\
 \text{and} & x_1, x_2 \geq 0
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ll}
 \text{Max.} & Z_1 = x_1 + 2 x_2 + 0 x_3 \\
 \text{S t ,} & 6x_1 + 8 x_2 + 0 x_3 \leq 48 \\
 & x_1 + x_2 - x_3 \leq 3 \\
 & x_1 + 0 x_2 + 0 x_3 \leq 4 \\
 & 0x_1 + x_2 + 0 x_3 \leq 3 \\
 \text{and} & x_1, x_2, x_3 \geq 0
 \end{array}$$

Now, we apply AHA simplex algorithm to find the optimal solution.

**Table 1. Initial AHA simplex table**

$x_1$	$x_2$	$x_3$		$b_1$
-1	-2	0	$\leq$	0
6	8	0	$\leq$	48
1	1	-1	$\leq$	3
1	0	0	$\leq$	4
0	1	0	$\leq$	3

The most negative coefficient in the first row is -2 which corresponds to the variable  $x_2$ . Hence the entering variable is  $x_2$ . Here minimum positive ratio is 3 which is coming at two positions. Select arbitrarily the fourth row. Therefore variable  $x_2$  enters into the fourth row. Pivot element is 1.

**Table 2. Intermediate AHA simplex table**

$x_1$	$x_2$	$x_3$		$b_1$
-1	0	0	$\leq$	6
6	0	0	$\leq$	24
1	0	-1	$\leq$	0
1	0	0	$\leq$	4
0	1	0	$\leq$	3

The most negative coefficient in the first row is -1 which corresponds to the variable  $x_1$ . Hence the entering variable is  $x_1$ . Here minimum positive ratio is 0, therefore variable  $x_1$  enters into the second row. Pivot element is 1.

**Table 3. Intermediate AHA simplex table**

$x_1$	$x_2$	$x_3$		$b_1$
<b>0</b>	<b>0</b>	<b>-1</b>	$\leq$	<b>6</b>
0	0	6	$\leq$	24
1	0	-1	$\leq$	0
0	0	1	$\leq$	4
0	1	0	$\leq$	3

The most negative coefficient in the first row is -1 which corresponds to the variable  $x_3$ . Hence the entering variable is  $x_3$ . Here minimum positive ratio is 4, which is occurring from the two positions. Hence we have to select it arbitrarily. Let variable  $x_3$  enters into the first row. Pivot element is 6.

**Table 4. Final AHA simplex table**

$x_1$	$x_2$	$x_3$		$b_1$
<b>0</b>	<b>0</b>	<b>0</b>	$\leq$	<b>10</b>
0	0	1	$\leq$	4
1	0	0	$\leq$	4
0	0	0	$\leq$	0
0	1	0	$\leq$	3

Now, it can be observed that all the coefficients of  $x_j$  in the objective inequality is either zero or positive. Therefore, this is an optimal solution. The optimal solution occurs at

$$x_1 = 4, x_2 = 3 \text{ and } x_3 = 4 \text{ with}$$

$$\text{Maximum } Z_1 = (4) + 2(3) = 10.$$

Similarly, we can find the values of other objective functions.

$$\text{For the second objective } x_1 = 4, x_2 = 3 \text{ with Maximum } Z_2 = 4$$

$$\text{For the second objective } x_1 = 4, x_2 = 3 \text{ with Minimum } Z_3 = (-2)(4) - (3)(3) = -17$$

$$\text{For the second objective } x_1 = 4, x_2 = 3 \text{ with Minimum } Z_4 = -3$$

The knowledge of these values are mandatory before applying the conversion techniques. That's why we have calculated.

Now reduced linear programming problem thus obtained by the techniques of Chandra Sen, Averaging techniques and New averaging techniques are being solved by our proposed Gauss elimination technique.

We get different bounded values for Z. Out of these  $Z = 3.9998$  is the only value that satisfies all the inequalities altogether. Hence  $Z = 3.9998$ . Now, putting  $Z = 3.9998$  into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting  $Z = 3.9998$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .



**Table 5. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using Chandra Sen's technique**

Objective Function	$\text{Max. } Z = \left(\frac{x_1+2x_2}{10}\right) + \frac{x_1}{4} + \left(\frac{2x_1+3x_2}{17}\right) + \frac{x_2}{3}$ $= 0.4676 x_1 + 0.7098 x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ <p style="text-align: center;">and</p> $x_1, x_2 \geq 0$
After first stage of elimination	<p style="text-align: center;">Max Z</p> $-1.10778443 x_2 + 12.8314799 Z \leq 48$ $0.51796407 x_2 - 2.13857998 Z \leq -3$ $-1.51796407 x_2 + 2.13857998 Z \leq 4$ $1.51796407 x_2 - 2.13857998 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 5.0358$ $Z \geq 3.9998$ $Z \leq 4.2587$ $Z \leq 3.9998$ $Z \geq 3.7408$

**Table 6. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using arithmetic averaging technique**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{7} - \frac{(-2x_1-4x_2)}{10}$ $= 0.4857 x_1 + 0.6857 x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ <p style="text-align: center;">and</p> $x_1, x_2 \geq 0$
After first stage of elimination	<p style="text-align: center;">Max Z</p> $-0.470660902 x_2 + 12.35330451 Z \leq 48$ $0.411776817 x_2 - 2.058884085 Z \leq -3$ $-1.411776817 x_2 + 2.058884085 Z \leq 4$ $1.411776817 x_2 - 2.058884085 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 4.4570$ $Z \geq 3.9999$ $Z \leq 4.1142$ $Z \leq 3.9999$ $Z \geq 3.8855$

We get different bounded values for Z. Out of these Z = 3.9999 is the only value that satisfies all the inequalities altogether. Hence Z = 3.9999. Now, putting Z = 3.9999 into the inequalities involved in second stage, we get different bounded values for variable x<sub>2</sub>. Out of these, x<sub>2</sub> = 3 is the only value that satisfies all

the inequalities altogether. Hence  $x_2 = 3$ . Now putting  $Z = 3.9999$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**Table 7. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using geometric averaging technique**

Objective Function	Max $Z = \frac{(2x_1+2x_2)}{6.324} - \frac{(-2x_1-4x_2)}{7.1414}$ = $0.5963 x_1 + 0.8763 x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$
After first stage of elimination	Max $Z$ $-0.817373805 x_2 + 10.0620493 Z \leq 48$ $0.469562301 x_2 - 1.677008217 Z \leq -3$ $-1.469562301 x_2 + 1.677008217 Z \leq 4$ $1.469562301 x_2 - 1.677008217 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 5.9889$ $Z \geq 5.0141$ $Z \leq 5.2578$ $Z \leq 5.0141$ $Z \geq 4.7704$

**Table 8. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using harmonic averaging technique**

Objective Function	Max $Z = \frac{(2x_1+2x_2)}{5.7143} - \frac{(-2x_1-4x_2)}{5.1}$ = $0.7421 x_1 + 1.13421 x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$
After first stage of elimination	Max $Z$ $-1.170273548 x_2 + 8.085163725 Z \leq 48$ $0.528378925 x_2 - 1.347527287 Z \leq -3$ $-1.528378925 x_2 + 1.347527287 Z \leq 4$ $1.528378925 x_2 - 1.347527287 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 8.1079$ $Z \geq 6.37103$ $Z \leq 6.80526$ $Z \leq 6.37103$ $Z \geq 5.9368$

We get different bounded values for Z. Out of these Z = 5.0141 is the only value that satisfies all the inequalities altogether. Hence Z = 5.0141. Now, putting Z = 5.0141 into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting Z = 5.0141 and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

We get different bounded values for Z. Out of these Z = 6.37103 is the only value that satisfies all the inequalities altogether. Hence Z = 6.37103. Now, putting Z = 6.37103 into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting Z = 6.37103 and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**Table 9. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using new arithmetic averaging technique**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{3.5} - \frac{(-2x_1-4x_2)}{3.5}$ $= 1.1428 x_1 + 1.7143 x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$
After first stage of elimination	Max Z $-1.000525026 x_2 + 5.250262513 Z \leq 48$ $0.500087504 x_2 - .875043752 Z \leq -3$ $-1.500087504 x_2 + .875043752 Z \leq 4$ $1.500087504 x_2 - .875043752 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 12.0009$ $Z \geq 9.7141$ $Z \leq 10.2858$ $Z \leq 9.7141$ $Z \geq 9.4124$

We get different bounded values for Z. Out of these Z = 9.7141 is the only value that satisfies all the inequalities altogether. Hence Z = 9.7141. Now, putting Z = 9.7141 into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting Z = 9.7141 and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

We get different bounded values for Z. Out of these Z = 9.81495 is the only value that satisfies all the inequalities altogether. Hence Z = 9.81495. Now, putting Z = 9.81495 into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting Z = 9.81495 and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**Table 10. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using new geometric averaging technique**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{3.4641} - \frac{(-2x_1-4x_2)}{3.4641}$ $= 1.1547 x_1 + 1.73205 x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$
After first stage of elimination	Max Z $-x_2 + 5.196154845 Z \leq 48$ $0.5x_2 - .866025807 Z \leq -3$ $-1.5x_2 + .866025807 Z \leq 4$ $1.5x_2 - .866025807 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 12.12434$ $Z \geq 9.81495$ $Z \leq 10.3923$ $Z \leq 9.81495$ $Z \geq 9.2376$

**Table 11. Solution by Gauss elimination technique of reduced LPP thus formed by MOLPP using new harmonic averaging technique**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{3.4285} - \frac{(-2x_1-4x_2)}{3.4285}$ $= 1.1666 x_1 + 1.7500x_2$
Constraints	$6x_1 + 8x_2 \leq 48$ $x_1 + x_2 \geq 3$ $x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$
After first stage of elimination	Max Z $-1.000514315 x_2 + 5.143151037 Z \leq 48$ $.500085719 x_2 - .857191839 Z \leq -3$ $-1.500085719 x_2 + .857191839 Z \leq 4$ $1.500085719 x_2 - .857191839 Z \leq 0$ $x_2 \leq 3$ $-x_2 \leq 0$
After second stage of elimination	$Z \leq 12.2507$ $Z \geq 9.9164$ $Z \leq 10.4993$ $Z \leq 9.9164$ $Z \geq 9.3328$

We get different bounded values for Z. Out of these Z = 9.9164 is the only value that satisfies all the inequalities altogether. Hence Z = 9.9164. Now, putting Z = 9.9164 into the inequalities involved in second stage, we get different bounded values for variable x<sub>2</sub>. Out of these, x<sub>2</sub> = 3 is the only value that satisfies all the inequalities altogether. Hence x<sub>2</sub> = 3. Now putting Z = 9.9164 and x<sub>2</sub> = 3 into the inequalities involved in

first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

The optimal solution of each reduced LPP thus formed by MOLPP discussed in numerical example by using AHA simplex algorithm and Gauss elimination technique is tabulated below:

**Table 12. Solution of MOLPP after applying AHA simplex algorithm and Gauss elimination technique**

<b>Chandra Sen's technique</b>	<b>Arithmetic average technique</b>	<b>Geometric average technique</b>	<b>Harmonic average technique</b>	<b>New arithmetic average technique</b>	<b>New geometric average technique</b>	<b>New harmonic average technique</b>
$x_1 = 4,$ $x_2 = 3$ and $Z = 3.9998$	$x_1 = 4,$ $x_2 = 3$ and $Z = 3.9999$	$x_1 = 4,$ $x_2 = 3$ and $Z = 5.0141$	$x_1 = 4,$ $x_2 = 3$ and $Z = 6.37103$	$x_1 = 4,$ $x_2 = 3$ and $Z = 9.7141$	$x_1 = 4,$ $x_2 = 3$ and $Z = 9.81495$	$x_1 = 4,$ $x_2 = 3$ and $Z = 9.9164$

It is obvious to verify that results are similar to the tabular values obtained from traditional simplex method or by Graphical method (in case of having only two variables) or by any other method available in the literature. Also it is very much clear from Table 12 that if we use new harmonic average technique to solve reduced LPP, then one can get the best optimal solution out of these techniques discussed above.

## 6 Conclusion

In this research paper, we used the AHA simplex algorithm and technique of Gauss elimination technique to solve MOLPP for the very first time. Different researchers used to solve traditional simplex method to solve MOLPP while we tried to apply different algorithms and techniques. It is observed that harmonic average technique possess dominance property than the techniques of Chandra Sen, Arithmetic averaging and Geometric averaging techniques. When we apply new statistical averaging techniques, the value of the objective function gets optimize as compared to statistical averaging techniques. New harmonic averaging techniques provide best optimize value of the objective function as compared to new arithmetic and new geometric averaging techniques.

## Competing Interests

Authors have declared that no competing interests exist.

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